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ABSTRACT

Twenty Algebra-Packets (A-PAKS) were developed by the investigator for technical education students at the community college level. Each packet contained a statement of rationale, learning objectives, performance activities, performance test, and performance test answer key. The A-PAKS condensed the usual sixteen weeks of algebra into a six-week period. An experimental group of 25 technical mathematics students completed the A-PAKS. Each member of the "traditional" group was selected from the total population of students taking technical mathematics during the years 1970-1975, and was matched to a student in the experimental group based on percentile scores on the standardized Hundred-Problem Arithmetic Skills Test. At the end of the A-PAK treatment, a standardized algebra test was administered to the experimental group and a student course-evaluation questionnaire was given. Results showed that the experimental group scored significantly higher ($p < .05$) on the algebra test than the "traditional" group. Results of the questionnaire showed that students liked the A-PAK procedure. Appendices include the A-PAKS, along with copies of the tests and the questionnaire given to the students. (DT)

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Condensing Algebra for Technical Mathematics

DONALD R. GREENFIELD

A MAJOR APPLIED RESEARCH PROJECT
PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF EDUCATION

NOVA UNIVERSITY

1976

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Abstract of a Major Applied Research Project Presented
to Nova University in Partial Fulfillment of the
Requirements for the Degree of Doctor of Education

CONDENSING ALGEBRA FOR TECHNICAL MATHEMATICS

By
Donald R. Greenfield

October, 1975

The instructional materials for this study were developed and the experiment designed in response to several identified needs of technical education students. The needs which were identified fell into several definite categories: (1) Many technical education students at the community college level experience difficulty with the algebra of numbers. This may be due to just one or a combination of several factors, such as a poor high school mathematical background, lack of recent number-algebra usage or need for a general review of arithmetic. (2) Technical education students--Machine Tool, Automotive, Electronics Technology, etc.--must be capable of solving complex algebraic equations that relate to their major field during the first semester of their technology program. (3) If deficient in algebraic skills, the first-semester student often experiences frustration; in fact, some students may even drop their course of study. (4) If,

however, algebra proficiency can be developed at a reasonable skills level early in the technical program, it is possible for the student to rephase into the formal college mathematics sequence. Should he desire, the student could possibly complete a course or two in the calculus of numbers.

A comprehensive review of the related literature was conducted. This review surveyed three areas directly relating to the study: (1) selection and condensation of key algebraic topics as identified by contemporary writers and educators, (2) individualized instruction and the preparation of instructional packages including an analysis of package components, and (3) studies utilizing individualized instruction in the presentation of algebra.

The following null hypothesis was postulated: Technical Mathematics students subjected to a method of condensed/systemized algebra would score comparable to traditionally treated students on a standardized algebra placement test.

In order to test the hypothesis and on the basis of the search of literature, twenty A-PAKS (Algebra Packets) were developed. Each packet contained a statement of rationale, learning objectives, performance activities (with an original treatise dealing with the mathematical concept embodied), performance test, and performance test answer key.

The A-PAKS were developed for a systemized presentation of a condensation of sixteen weeks of algebra into a six-week period. The A-PAKS were subsequently tested with an experimental Technical Mathematics group consisting of twenty-nine students at the College of Marin. Twenty-five of the original subjects completed the experiment in approximately six weeks of treatment.

A Hundred-Problem Arithmetic Skills Test was administered to the experimental group, and on the basis of this test a paired traditional group of Technical Mathematics students was assembled. At the end of the A-PAK treatment, the algebra test was administered and the results were subjected to a statistical analysis consisting of an F-test for significance, means and median score analysis, and frequency polygon comparison. A like and dislike effect of the treatment was determined by means of a subjective student questionnaire.

The treatment was determined to be significant at the pre-stipulated 5 percent (.05) level of confidence, and the null hypothesis was rejected. The experimental group scored significantly better on the placement test than the traditional group even though they had scored lower than the traditional students on the basic arithmetic skills test. The experimental group exhibited a convincing like effect of the treatment.

The study concluded with the researcher's summary and suggestions for more experimentation at all school levels with the phasing of mathematics courses and their empirical structure in the endeavor to provide mathematical capability when it is most needed.

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CHAPTER I

STATEMENT OF PURPOSE AND BACKGROUND

The purpose of this Major Applied Research Project (MARP) was to develop a systemized condensation of a traditional algebra course into a six-week (18 hour) presentation. An integral part of the MARP was the on-site testing of the product with a Technical Mathematics class at the College of Marin.

At the College of Marin there are several Vocational-Technical degree programs that require of their students at least one college course of Technical Mathematics, while a few of the programs demand additional mathematics--an even higher level of mathematical proficiency. Either Technical Mathematics, General Mathematics, or a similar course is typically offered and required of technical majors at most of the California community colleges.

The course content of Technical Mathematics is usually review in nature, wide in scope, and encompasses a treatment of arithmetic, algebra, geometry, and practical trigonometry. Historically, Technical Mathematics was instituted at the College of Marin in response to the demand by the technical education instructors for a comprehensive review of the mathematics traditionally taught at the high school level. The course currently encourages its enrollees--primarily technical students--to realize two extremely important objectives.

It is desirable that the student of Technical Mathematics:

1. Solve mathematical problems related to his specific technical area as the student simultaneously learns theory and manipulative skills.

2. Learn the necessary mathematical skills and/or prerequisites in order for the student to re-enter the formal college mathematics sequence should he so desire.

The latter objective has incidentally magnetically attracted some nontechnically oriented students. They have elected the course essentially for the purpose of "brushing up" skills prior to enrolling in the formal college mathematics sequence--Advanced Algebra, Trigonometry, Calculus.

Both students who are required to complete Technical Mathematics and others who simply elect the course to improve their skills are by and large not particularly proficient in higher mathematics; in fact, many of these students score in the lowest quartile in basic arithmetic skills. However, these same students are expected to solve relatively complex calculations in their respective major areas of technical concentration. The following problem will demonstrate the level of proficiency necessary during the first semester of Machine Tool Technology at the College of Marin:

> A chord table can be computed by the following method and utilized for spacing on the circumference of parts preparatory to machining.
>
> N = number of divisions required
>
> x = chord length
>
> x = 2 times side c
>
> Angle $C = \frac{\frac{360}{N}}{2}$
>
> $C = 30^\circ$
>
> Angle $A = 90^\circ$
>
> Angle $B = 180 - (A + C)$
>
> $B = 180 - (90^\circ + 30^\circ)$
>
> $B = 60^\circ$
>
> Side a = radius
>
> $a = .500$
>
> Side c = a times sine C
>
> c = .5 times sine 30°
>
> c = .5 times .5
>
> $c = .250$
>
> Side x = 2 times side c
>
> x = 2 times .25
>
> $x = .5$ [1]

---

[1] Donald R. Greenfield, "Spacing on the Circumference," Industrial Arts & Vocational Education/Technical Education Magazine (May, 1968), pp. 64-68.

This example is a typical machine shop layout problem--trigonometric in design. It can be adequately treated in the Machine Tool lecture by the technical instructor; however, in order to solve spacing problems when given the technique, the student must obviously possess an algebra capability that the instructor assumes each student has achieved since the instructor cannot devote time to teach algebra.

Similar problems for the technical areas of Automotive, Drafting, and Electronics Technology demand algebra proficiency. How, then, is the technical student able to solve these problems the first semester if he should lack the algebra capability? Far too many technical students cannot; consequently, they suffer frustration and discouragement.

Technical Mathematics courses devote the lion's share of their sixteen weeks of course content to remedial arithmetic and algebra. A traditional algebra section also requires approximately sixteen weeks. Could this sixteen weeks of material (algebra) be sieved for its essentials--the most important concepts--and then might the material be adequately covered in the first six weeks of Technical Mathematics? If so, this procedure would provide almost immediate capability for the problem solving in the major technical areas. The remainder of Technical Mathematics time would then be profitably devoted to the consideration of needed more advanced concepts--logarithms, coordinate geometry, vectors, etc. The arithmetic review is encompassed by performing actual computations, i.e., practice.

There is another facet of the problem that is similarly concerned with time--that most valuable of student commodities. It is extremely desirable that selected areas of the Vocational-Technical cluster--namely, Machine Tool, Drafting, Electronics, and Engineering--encourage the students to achieve as high a degree of the formal mathematics as they are capable of realizing. Unfortunately, under the current college mathematics structure, the technical student must spend one semester in each of the following as per the traditional, empirical nature of mathematics:

Technical Mathematics)
Algebra) Total time:
Advanced Algebra) eight semesters
Trigonometry) or four years!
Calculus I, II, III, IV)

It is apparent that in adhering to the above sequence the technical education student will at best progress through the trigonometry level in his four semesters at the community college. This leaves very little margin for failure or half-success. For should the student fail to receive a solid "C" grade in any of the courses, he is not allowed or even encouraged to proceed. While it is possible to hurdle a course, by and large, it is almost impossible for the technical student to reach the calculus level unless he enters the college already well qualified. If, however, remedial algebra time is shortened in Technical Mathematics, higher concepts can be treated, thereby improving the student's chances for a better placement test score and consequently a higher entrance level--perhaps at the advanced algebra or trigonometry level.

Technical Education students at the College of Marin are encouraged to enroll in courses of higher mathematics not only for skills applicable in their respective areas but for future upgrading of their vocational potential. The calculus is the magic key that opens the door leading to higher-paid and more satisfying vocations and professions. Even those students who only complete one semester of calculus aspire to, and many eventually achieve, advancement either on the job or by means of additional and/or continuing education.

CHAPTER II

REVIEW OF RELATED LITERATURE

Three distinct areas of literature were surveyed for purposes of this experiment. Each of these areas is summarized under "Review of Related Literature"; however, for clarity of presentation, some of the methodology resulting from the review is treated under more apropos headings. (An example is the product development that is treated under "Sources of Information and Assistance.") The three areas investigated include:

Review 1. A review of algebra materials with special focus on the selection and condensation of the most important algebraic concepts.

Review 2. An investigation of expository publications that deal with the components of individualized instructional materials and the preparing of instructional packets.

Review 3. A search of the literature for data pertaining to individualized instruction in algebra. This included publications, studies, and theses.

Review 1: Review of Materials for Selection and Condensation of Algebraic Content for the Experiment

There is a plethora of information published about innumerable concepts that are recommended for a first-semester course of college algebra (as well as Technical Mathematics), but surprisingly few authors advocate any degree of refinement or condensation. Two well-known mathematicians have, however, presented a stimulating first-semester course content in thirteen key chapters. They state that "the actual amount of material covered and the depth of coverage will depend to a large degree upon the background of the students involved."[2] They further maintain that with some modification--eliminating optional sections and de-emphasizing certain topics--their course of college algebra could be covered in one quarter. This material, interestingly, includes behavioral objectives as well as innovations like "flow charting."

Selby has similarly presented twelve algebra topics consisting of a bridge from arithmetic through first-semester algebra by utilizing a practical approach.[3] Selby defines "practical" as referring to an emphasis on the techniques of problem solving. The topics (chapters) of material are broken into short steps called "frames," and the student is encouraged to progress at his own pace. Both objectives and self-type tests are incorporated in a combination self-study and classroom-use approach.

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[2] Mary P. Dolciani and Robert H. Sorgenfrey, Elementary Algebra for College Students. See also Mary P. Dolciani and Robert H. Sorgenfrey, Instructors Guide and Solutions.

[3] Peter H. Selby, Practical Algebra.

A similar model is advocated by Heywood who supplies five key units, although each unit is subdivided into several frames.[4] The units embody many concepts that are woven into an extremely comprehensive treatment of algebra. Heywood's program allows for three possibilities:

1. A lecture-tutor procedure that can provide the transition from the student-centered classroom to the college-lecture format.

2. A class that requires only a review, with emphasis on difficulties as they are experienced individually.

3. A completely self-contained course in elementary algebra for self-instruction.

Technical Mathematics is treated very comprehensively by Boyce, Slade, and Margolis in their recently revised textbook, but they devote only one of twenty-two chapters to practical algebra.[5] This is about twenty pages. The following topics are sparsely covered: use of letters, negative numbers, substitution, addition, subtraction, multiplication, division, signs of grouping, equations. This approach appears to represent the most minimal algebra treatment; few Technical Mathematics instructors would be amenable to this sparse treatment. Further, the authors infer that algebra is by and large already known and an elementary review will suffice. The researcher believes this "overly condensed" version of algebra serves little practical purpose other than to confuse students who are in need of skill development.

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[4] Arthur H. Heywood, Elementary Algebra: Lecture-Lab, p. iii.

[5] John G. Boyce, Samuel Slade, and Louis Margolis, Mathematics for Technical and Vocational Schools.

The product of a five-year project--The Milwaukee Area Technical College Mathematics Series--was designed for a two-semester Technical Mathematics course. According to the authors, after five years had elapsed the following had become evident:

> 1. The dropout rate in the course has been reduced by 50%.
> 2. Average scores on equivalent final exams have increased from 55% to 85%.
> 3. The rate of absenteeism has decreased.
> 4. Student motivation and attitudes have been very favorable.[6]

Field tests with over 4,000 students have enabled constant revision. At the college level, this system can be used in pre-technical, apprentice, trade, and developmental programs. The authors emphasize:

> Though not designed for students with serious deficiencies in arithmetic, it has been successfully used with students who were below average in arithmetic skills.[7]

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[6] Thomas J. McHale and Paul T. Witzke, Basic Algebra, pp. v-vi.

[7] Ibid.

The series consists of four volumes: Basic Algebra, Calculation and Slide Rule, Basic Trigonometry, Advanced Algebra. The three components of the system are: (1) the four textbooks, (2) a diagnostic assessment--essentially test booklets, and (3) a teacher who serves as manager of the system.

The first two sections of the Milwaukee series, "Algebra" and "Slide Rule," undoubtedly comprise the first semester, with probably two-thirds of the semester (about twelve weeks) devoted to algebra. The algebra portion contains ten chapters with each chapter subdivided into an average (median) of fifteen subclassifications. This material is organized so that several variations of a concept are treated as subclassifications, thus presenting an extremely in-depth approach! The system appears very flexible, suitable for a conventional classroom text in a learning center or as an adjunct--supplementary to another textbook. Its main drawback, for purposes of the researcher's experiment, is the one hundred fifty concepts (ten chapters times fifteen subclassifications per chapter) albeit many of the subclassifications are unimportant or not even concepts but adjuncts.

An interesting departure from the conventional Technical Mathematics approach has been developed by

Bailey.[8] The system commences with a liberal arts orientation of the arithmetic fundamentals and rapidly advances into algebra and trigonometry. There is a core text and a series of accompanying laboratory modules that provide an individualized correlated study between mathematics and the student's technical area. There are six technical areas encompassed by the system:

1. Electronics Technology.
2. Machine Shop Technology.
3. Automotive Technology.
4. Mining Technology.
5. Drafting Technology.
6. Liberal Arts Education.

Bailey's system is the result of three years of concentrated research involving both industry and two-year colleges. In the algebra portion of the system, which appears relatively structured, there are about ten key areas (i.e., powers and roots, formulas, simultaneous equations, etc.) that are tied to eleven modules in the drafting and machine shop correlation modules text. No time frame is specified since the system is eventually individualized combining both lecture and independent study. However, it appears that the elementary algebra portion might be adequately encompassed in a maximum of eight weeks.

Bailey's modules might not be particularly adaptable for California community college technical programs since there is a wide diversity in specific teacher-selected technical curriculum, but the approach is by far one worthy of careful analysis.

---

[8] F. A. Bailey, Competency Based Vocational-Technical and Liberal Arts Mathematics, and Correlated Mathematics for Drafting and Machine Shop Technology.

Sperry's Programmed Algebra is representative of the "programmed model" where the student progresses from frame to frame--very sequential, advancing after frame mastery at his own pace.[9] Some advocates of this model, including Sperry, note that a few students seldom attend class but are nevertheless successful. With this programmed material, it is assumed that most of the students are seeking a review.

There is no attempt at condensation; some twenty-nine units are presented within a time frame of a full semester. While the lessons are excellent and the progression appears very logical, this type of approach does not lend itself to Technical Mathematics where more forceful teacher direction appears to better meet student needs.

About 1960, mathematical educational materials underwent some very close scrutiny by noted mathematicians and educators. It was felt some selected mathematical ideas should receive more interesting and understandable treatment. A series of volumes under the "New Mathematical Library" cover was the result.

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[9] J. Bryan Sperry, Programmed Algebra.

Topics not usually included in the traditional school mathematics curriculum were treated by professional mathematicians. The project was titled "Monograph Project of the School Mathematics Study Group" (SMSG). This group represented all facets of the mathematical profession, and the chief aim of SMSG was stated as the improvement of teaching mathematics in the schools.

SMSG was headquartered at Stanford University in California and spearheaded the revitalization of mathematics at all levels of education. Most of the contemporary modern mathematics trend is a result of SMSG influence. Four of the several books in the SMSG collection contain concepts and treatment that are directly applicable to the course content for this experiment.[10] Consequently, some of the researcher's algebra materials were influenced by the historical but still very pertinent and well-appraised SMSG publications.

In summary, a preliminary search of the literature for a time frame of condensation as well as some specific materials needed for developing the product (curriculum) for the experiment yielded at least one attractive direction. It was the researcher's intention to employ those concepts that have been consistently underscored by the leading mathematicians and writers. While a minimum of research and publication has been focused on compressing the total time interval for algebra mastery, Dolciani[11] and Bailey have at least opened a preliminary door for the possibility of condensing both time and algebra material.[12] There was then, the researcher believed, a need for this kind of an experimental approach--particularly for technical education students.

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[10] Edwin Beckenbach and Richard Bellman, An Introduction to Inequalities.
Philip J. Davis, The Lore of Large Numbers.
Ivan Niven, Numbers: Rational and Irrational.
Leo Zippin, Uses of Infinity.

The selectivity employed in the product, it appeared, was crucial to the experiment. The researcher, when selecting, had not only to refer to the available literature but also depend upon his personal expertise based upon fifteen years of teaching both theoretical and applied mathematics to both remedial and gifted students.

Review 2: Preparing Individualized Instruction Material

In their report, Postlethwait and Russell identify the characteristics of minicourses--self-contained instructional packages that deal with a single conceptual unit of subject matter.[13] Some of these characteristics include a combination of learning experiences, the definition of the teacher's role as a resource person and motivator, specified performance objectives, test items to measure objective mastery, and students proceeding at their own rate. Particularly germane to this research was the "flexibility" characteristic: "Minicourses can be structured into a greater variety of patterns consistent with different approaches or themes."[14] The researcher utilized the minicourse structure with a specified total time for the experiment of one-third of a regular semester's time. The major conceptual unit in this case was algebra; however, approximately eighteen subconcepts were packaged.

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[11] Dolciani, op. cit.
[12] Bailey, op. cit.
[13] S. N. Postlethwait and James D. Russell, *Minicourses - The Style of the Future*.

Postlethwait and Russell further state:

> Having a design goal and an evaluation plan, the minicourse developer is able to correct faulty instructional materials and to know when he has succeeded in developing a successful minicourse.[15]

Perhaps this aspect of the minicourse is the most valuable for the teacher/experimenter: it is possible for him to keep his courses meaningful through continual student and personal evaluation.

An instructional system at the operational level, according to Herrscher, is a map depicting the path available to the learner as he presses for mastery of a selected unit of instruction.[16] Herrscher diagrams this system (see Figure 1) to represent the general

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[14] Postlethwait and Russell, op. cit.
[15] Ibid.
[16] Barton R. Herrscher, Implementing Individualized Instruction.

An INSTRUCTIONAL SYSTEM

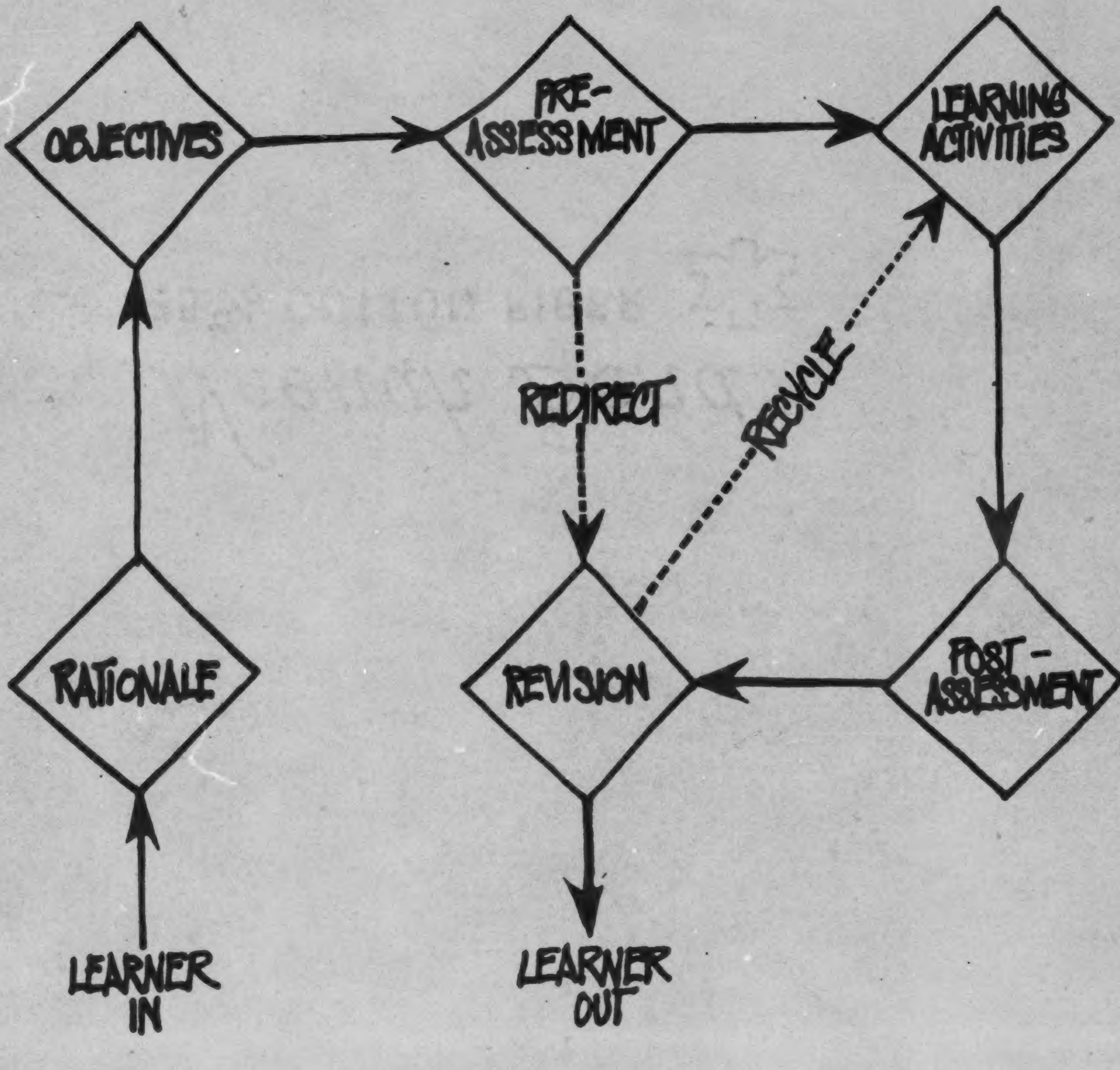
OBJECTIVES
PRE-ASSESSMENT
LEARNING ACTIVITIES
REDIRECT
RECYCLE
RATIONALE
REVISION
POST-ASSESSMENT
LEARNER IN
LEARNER OUT

Figure 1

model for a systems approach. Note especially the learner's option at the pre-assessment juncture. He can either exit from the system or direct himself to the learning activities. Herrscher further maintains that a course composed of a series of self-instructional units is highly individualized and yet allows for uniform instruction of a large number of students on an individual basis.

There are several alternatives for preparing units of instruction. Since a system is composed of components, and in order to remain in focus on the unit concept in total, some plan of development consistency is almost mandatory. Johnson and Johnson specify six stages of development in the following order:

1. Identifying prerequisites.
2. Objectives.
3. Criterion measures.
4. Learner activities.
5. Related content.
6. Media and materials.[17]

Stuart and Rita Johnson also emphasize that there are at least two ways of sequencing the instructional environment: (1) arranging the learning material-content, and (2) arranging the learner's activities or behavioral responses.

Botts and Reed identify three distinct methods of course organization:

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[17] Stuart R. Johnson and Rita B. Johnson, Developing Individualized Instructional Material.

1. Common Learnings Plus Individualized-Content Tracks (objectives for each learning and a series of proficiency tracks).

2. The Continuous Progress Method (a series of behavioral objectives each identifying an instructional package).

3. The Three-Track Method (three levels of learning activities in each package).[18]

A more pragmatic method of developing programs is advocated by Jacobs wherein four basic stages are observed:

1. Specifying goals.
2. Writing frames.
3. Trying out and revising.
4. Validation.[19]

The fourth stage is the key to the entire proposition and Jacobs advocates:

> The program should be validated, or tested before it is released for general school use. In validation one determines how much students learn from the program. A standardized test, or a test constructed for the purpose, is used to assess learning. . . .[20]

Jacobs was an early proponent of individualized instruction, leaning heavily on a "programmed model." His emphasis on testing in a formal statistical fashion greatly influenced the development and testing of learning systems.

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[18] Robert E. Botts and Donald R. Reed, Individualized Instruction.

[19] Paul I. Jacobs, Milton H. Maier, and Lawrence M. Stolvrow, Guide to Evaluating Self-Instructional Programs, p. 11.

[20] Ibid.

Components - rationale. Rationale sets the stage--the tone of the unit of material. According to Herrscher, the content of each unit of the program must be selected on the basis of its relevance to both the student's needs and the institution's purpose.[21] Botts and Reed employ rationale as the "purpose"--sometimes called the goal or goals of the instructional package.[22] The goals may be all-encompassing, embracing the entire course, or they might pertain solely to each individual unit. Some descriptive names for rationale are suggested by Kapfer who utilizes such terms as component idea, skill, or attitude.[23] It appears to be the general consensus that some statement, be it purpose, goal, or rationale, helps to define the target for the student and provide some cohesiveness to the unit of instruction.

Components - objectives. There has been a plethora of professional publications pertaining to the writing of "objectives," one of the key components of an instructional system. Only a few comments are presented here

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21. Herrscher, op. cit.
22. Botts and Reed, op. cit.
23. Philip G. Kapfer, Practical Approaches to Individualizing Instruction. (As cited in John E. Roueche and Barton R. Herrscher, Toward Instructional Accountability, p. 33.)

since there is much more uniformity than difference amongst the supporters of educational objectives.

Robert F. Mager, one of the early advocates of instructional objectives, aptly pinpoints and defines "an objective" as ". . . a description of a pattern of behavior (performance) we want the learner to be able to demonstrate."[24] Mager further qualifies objectives as denoting measurable attributes that are observable: that an instructional objective describes an intended outcome. It is stated in behavioral, or performance terms, and communicates the instructional intent of the person selecting the objective.

Ogletree and Hawkins concur with Mager and also emphasize that objectives should be stated in terms which will indicate the kinds of change (behavior) to be brought about in the learner.[25] Objectives may reside in the cognitive, affective, and psychomotor domains.

Another enlightening dimension is added when objectives are composed, according to Hernandez, using a "desired behavior" and "a competence level method."[26] Both these aspects are incorporated into every objective.

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[24]. Robert F. Mager, Preparing Instructional Objectives, p. 3.

[25]. Earl J. Ogletree and Maxine Hawkins, Writing Instructional Objectives and Activities for the Modern Curriculum.

[26]. David E. Hernandez, Writing Behavioral Objectives.

Behavioral objectives must be selected and written with care for there are some concealed instructional dangers that could work to the student's disadvantage. McAshan identifies five potential dangers.[27]

1. Students are different in learning.
2. Teachers are different in selecting objectives.
3. Too much standardization in the objectives may result in the loss of teacher flexibility.
4. Mediocre objectives due to time limitations or lack of teacher training may result in mediocre results.
5. Objectives must be continually evaluated and restated if they are to remain appropriate.

Objectives then appear to be critical and therefore much care must be taken in their selection and phrasing.

Components - performance activities. Performance activities or learning activities are the means whereby the student does "things" in order to realize the objectives. Some typical learning activities include:

1. Readings, i.e., articles, books, etc.
2. Practice sets, i.e., mathematical problems.
3. Making things, i.e., industrial arts projects.
4. Viewing/listening, i.e., audiovisual.

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[27] H. H. McAshan, Writing Behavioral Objectives, A New Approach.

Some teachers have attempted to combine individualized instruction with the lecture method.[28] They cite that the "multimedia approach" for activities may overlook more complicated but very important learnings in mathematics, like proof or heuristics, which can be successfully treated in a variation of the traditional lecture. In fact any activity, if meaningful, can be a desirable learning activity--even attending the traditional lecture or chalkboard talk.

Of increasing contemporary importance is the Learning/Resource Center. Berg, Metcalf, and Gravely envision a multimedia spatial center containing areas for reading, individual study, conference and/or seminars, audiovisual, materials preparation, printed and nonprinted materials, reference catalogs, and audiovisual preparation and staff workrooms.[29] At the College of Marin, there is an outstanding "Center for Independent Study" with excellent facilities and resources.

Pre-test and post-test. The pre-assessment in a system should precede formal instruction. Based on the objectives, this helps to determine three important facts aptly summarized by Herrscher.[30]

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[28] Harold L. Schoen, "A Plan to Combine Individualized Instruction With The Lecture Method," The Mathematics Researcher (November, 1974).

[29] Lyle L. Berg, Elinor Gay N. Metcalf, and Andree G. Gravely, Individualization of Instruction, The Role of the Learning/Resource Center.

[30] Herrscher, op. cit.

1. Has the student the prerequisite capabilities?

2. Does the student already possess the behavior as specified by the objectives?

3. Where should the student be placed in a graduated learning sequence if he possesses *some* of the specified behaviors?

Since this experiment was concerned primarily with review materials and the subjects historically have been experiencing mathematical difficulties, the pre-test was dispensed with in the experiment.

The post-assessment, according to Herrscher, is best if it is criterion-referenced as differentiated from norm-referenced; that is, the testing is used to determine performance with respect to a criterion (objectives).[31] Further, if adapted to individual learning rates, instruction allows for objective mastery concept in grading. The testing should assess the teaching. Aptitude is viewed as the *time* required to master objectives.

A valuable adjunct of the criterion-referenced post-test is that it *measures learning* (and teaching). No learning does not mean an "F." It merely implies that the student should recycle back into the system.

In summary, there is fairly general agreement as to what may comprise packages of individualized instruction as well as methods of application. The researcher, on the basis of this review and experience, has selected key components. Cycling of students in the system was based upon the guidelines exposed by the search. (See Figure 2.)

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[31] Herrscher, *op*. *cit*.

Review 3: Review of the Literature Pertaining to Individualized Instruction in Algebra

A search of ERIC has yielded several studies and writings that bear upon this experiment. These materials have been critically screened, and only the most pertinent have been summarized for this review.

In 1965, a programmed course in updated algebra called Modern Mathematics was tested at the high school level with a class of twenty-two members.[32] One of the specific objectives of the study was to offer more individualized study since two problems of the school were identified as: (1) small class enrollments and (2) providing for children of varying abilities. The teacher gave a zero for pre-test scores and composed a final test (post-test) at the end of the year. The mean post-test score was 48.82 and is unadjusted since several students did not complete all the work. The instructor summarizes that the students learned a significant amount from the program. The gain-ratio revealed that students learned 50 percent of what they could possibly have learned.[33] The instructor also emphasized that, at best,

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[32] ERIC ED 020 063. Mary Joe Clendenin, Programmed Mathematics.

[33] Ibid.

ALGEBRA INSTRUCTIONAL SYSTEM

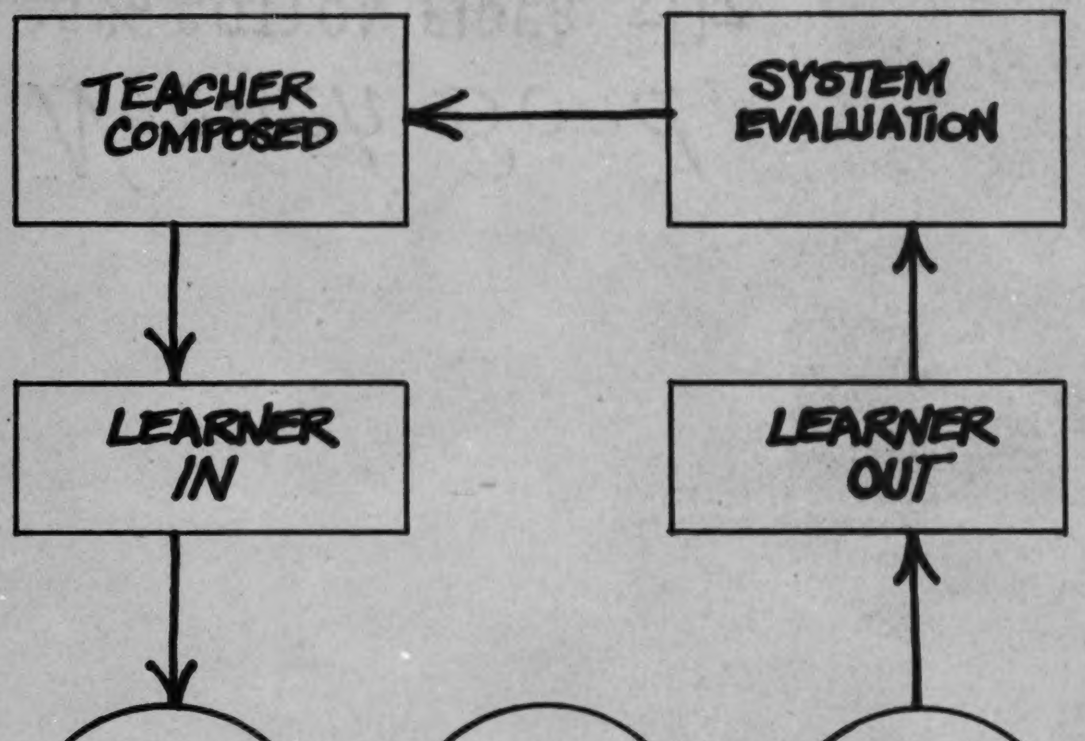
TEACHER COMPOSED
SYSTEM EVALUATION
LEARNER IN
LEARNER OUT
RATIONALE
PERFORMANCE TEST
TEST EVALUATION
RECYCLE
INSTRUCTIONAL OBJECTIVES
PERFORMANCE ACTIVITIES

Figure 2

programmed instruction was seen as the individualization of instruction.

Samples of instructional objectives applicable for a community college intermediate algebra course have been compiled by Purdy.[34] These objectives are offered as samples that correspond to skills, abilities, and attitudes that mathematics instructors may want their students to acquire. The author presents an intermediate algebra course that contains two of the components of a traditionally packaged course: (1) the objectives of each unit and (2) the self-test for each unit. Of significance is the number of units--twenty-one. Thus, this semipackaged course contains only three more units than the one prepared by the researcher for this experiment. Most of the objectives presented are in the cognitive domain, geared toward the realization of the variables of knowledge, comprehension, and application. This type of a treatment is somewhat traditional when the subject matter is mathematical.

Another evaluation of an individualized program for ninth-grade algebra and basic mathematics was presented by Clinton Ludeman and others.[35] The two primary objectives of the program were to increase achievement and create a positive attitude toward mathematics. Both a needs questionnaire and an attitude questionnaire were administered.

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[34] ERIC ED 067 077. Leslie Purdy, Instructional Objectives for a Junior College Course in Intermediate Algebra.

[35] ERIC ED 086 545. Clinton Ludeman, et al, Final Evaluation Report, Project Video-Tape Packages Mathematics.

The algebra achievement gains of thirty-one experimental students were compared to control students in the previous year's regular classes; no significant differences were found. Under t-score treatment (after twenty weeks' instruction) the experimental mean algebra score was 30.50 as compared to 26.90 for the control group. Thus the experimental group was only about four points higher than the control group. The t-statistic was 1.858. At the 95 percent confidence limit and with 29 degrees of freedom, the value of t for significance is 2.045 or higher. Thus no significant difference was indicated. Interestingly, however, 65 percent of the experimental students showed a more positive attitude toward mathematics after one year of exposure to the program. Put succinctly, the program met the challenge posed to it in terms of its objectives.

The program can be considered a successful attempt to present a form of individualized instruction in the mathematics program in order to help the slow learner as well as the advanced student realize his full potential.

The novel approach of having student assistants develop and direct the instruction was employed by

Dr. Cecil W. McDermott.[36] An analysis of covariance design yielded the conclusion by Dr. McDermott:

> . . . that college students capable of directing their own learning can successfully master intermediate algebra concepts when the students are placed in a well-defined self-pacing audio-tutorial learning system.[37]

Although the experiment was conducted at the four-year college level, the concepts treated were remedial-intermediate algebra. McDermott also recommended that learning systems be developed and used for teaching of low-level mathematics concepts; further, that each college should develop its own learning systems. Fourteen units (packages) were used in the study; F = 1.01; the null hypothesis was accepted at .01 level. Thirty students composed the experimental group; twenty-seven, the control group.

A manual for the preparation of programmed text material based on School Mathematics Study Group work and materials was prepared in 1961 and subsequently revised (1961-64) as a result of a project under the direction of a noted panel coordinated by Leander W. Smith, Assistant to the Director of SMSG, at Stanford University.[38] The researcher followed some of these guidelines in preparing A-PAK materials since his presentation utilized an SMSG approach.

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[36] ERIC ED 077 774. Cecil W. McDermott, The Development of an Individualized Learning System for Students Studying Intermediate Algebra, Final Report.

[37] Ibid.

[38] ERIC ED 022 688. W. G. Chinn and others, School Mathematics Study Group Report No. 1, The Programmed Learning Project.

A set of ten Learning Activity Packages (LAPs) in beginning algebra and nine in intermediate algebra prepared by Diane Evans appear to most closely align with the researcher's projections for his A-PAKS.[39] Evans' LAPs are prepared for the high-school level; and while the content and approach are similar to the researcher's, the level is unfortunately inappropriate for the community college.[40] Nevertheless, Evans' work does indicate the degree of contemporary awareness of the possibilities of a systems approach toward learning mathematics concepts.

Bill Holland has prepared similar packaged materials which deal with advanced algebra and trigonometry.[41] His materials are identical in format to Evans' and a continuum of her work. Similarly, the package components include rationale, behavioral objectives, activities (reading assignments, problem sets, tape recordings, filmstrips), self-evaluation problem sets, and references for further work.

The rationale employed by both Evans and Holland is indicative of the quality of their LAPs and does aptly unify the concept or concepts of each of the units.

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[39] ERIC ED 069 504. Diane Evans, Learning Activity Package, Algebra.

[40] Refer to A-PAKS included in this study.

[41] ERIC ED 069 505. Bill Holland, Learning Activity Package, Algebra-Trigonometry.

For his Programmed Math Continuum, Harrigan prepared a handbook to accompany a series of programmed study guides for first-year algebra.[42] Of particular note is the emphasis placed upon one of the components of the continuum. Harrigan outlines the chronological sequence that should be followed by the writer and emphasizes that the writer should give alternate explanations of difficult concepts.

The researcher also employed this approach when preparing his A-PAKS. While Harrigan's technique is not original, it is seldom encountered in systemized instructional packets. The package writer can call upon his teaching expertise and experience for preparing this expository component.

There have been several additional research projects directed toward preparing algebra courses that are structured in a systemized, self-study fashion; however, these projects incorporate computer-assisted instruction into their systems. The New York Institute of Technology has developed a noted series[43] as have Harrigan[44] and

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[42] ERIC ED 075 200. J. Ward Harrigan, Programmed Math Continuum, Level One, Algebra, Handbook.

[43] ERIC ED 075 206. New York Institute of Technology, Programmed Math Continuum, Level One, Algebra, Volume 5.

[44] Harrigan, op. cit.

Love.[45] And, while the computer-assisted instruction has much to enrich and facilitate a full-blown algebra course involving many students, it is not particularly adaptable to a one-semester class of limited size.

A variation of logical versus scrambled instructional sequences generally reports negative or no-difference findings according to Spencer and Briggs.[46] This aspect of instructional systems has been heavily investigated previously, and the researcher assumed his instructional sequencing of the A-PAKS would not necessarily add a variable that would unduly influence the outcome of the experiment.

In summary, a search of the literature dealing with instructional packages of mathematics, their development, and testing has brought to light the following:

1. Instructional packages are not new in the field of mathematics education at either the secondary or college levels.

2. Instructional packages are recommended by some researchers for pre-calculus mathematics.

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[45] ERIC ED 034 403. William P. Love, Individual Versus Paired Learning of an Abstract Algebra Presented by Computer-Assisted Instruction.

[46] ERIC ED 063 807. Rosemary V. Spencer & Leslie J. Briggs, Application of a Learning Hierarchy to Sequence an Instructional Program, and Comparison of This Program with Reverse and Random Sequences.

3. Excellent guidelines have been developed for programming SMSG materials and writing pertinent objectives.

4. The statistical studies have indicated that the null hypotheses (no significant difference between experimental--packages--and control--traditional--groups) is the probable result. Other effects such as more individualized instruction or better reception by the students can be anticipated with a systems approach.

5. Experimentation with the condensing of time for algebra concept mastery has not been located in the preliminary search of the literature; therefore, this experiment apparently is unique in this respect.

CHAPTER III

STATEMENT OF HYPOTHESES

Eighteen original instructional units were tested with a Technical Mathematics class at the College of Marin. The researcher hypothesized null: Technical Mathematics students subjected to a method of condensed/systemized algebra would score comparable to traditionally treated students on a standardized algebra placement test.

The alternate hypothesis was two-tailed: either the experimental group would score higher or the traditional group would score higher on the standardized algebra placement test.

CHAPTER IV

SOURCES OF INFORMATION AND ASSISTANCE

Development of the A-PAKS

The A-PAKS were designed and prepared by the researcher solely for this research project. Both the design of the material and the algebra topics that were selected for inclusion were chosen as a direct result of the data summarized under "Review of the Literature." The researcher was greatly influenced by the two following resources for his A-PAK design and composition.

1. During the Fall Semester of 1972 an educational workshop was carried out at the College of Marin by Professors James Staley (Engineering-Electronics) and Jay Stryker (Electronics). The main thrust of this twenty-hour workshop was to encourage and assist selected College of Marin teaching staff in the preparation of individualized instructional packages. The researcher was selected as a participant and was fortunate in having his work critiqued by both Professors Staley and Stryker (each participant prepared a comprehensive module of instruction).

2. The Nova Module "Curriculum" was presented to the Nova San Francisco Cluster in January, 1974.[47]

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[47] Barton R. Herrscher, Dayton Roberts, John Roueche, Curriculum Development.

Dr. John Roueche served as the national lecturer. Dr. Roueche, a recognized leader on contemporary educational change, required of each cluster member (the researcher is a member of the San Francisco Cluster) the preparation and on-site testing of an instructional package (individualized instruction).[48] Under Dr. Roueche's guidance, the researcher's conceptualization of instructional packages was crystalized and, as a result of that preliminary exposure, refined.

The technique of composing an original technical treatise or monogram for each of the algebra topics compiled in the A-PAKS was elected by the researcher in order to incorporate into the system his expertise amassed over a period of several years of teaching algebra at both the secondary and college levels of education.

Twenty A-PAKS were composed and assembled.[49] The topic titles include:

A-PAK Orientation
A-PAK #1, Operational Order and Number Properties
A-PAK #2, Fundamental Operations - Addition and Subtraction
A-PAK #3, Exponents

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[48] Dr. Roueche is a professor at the University of Texas at Austin.

[49] A complete A-PAK text assemblage is included in Appendix A since it was the tangible "product" of this research project. Although the material is technical, it presumably would be of interest to mathematics teachers and/or educational researchers.

A-PAK #4, Fundamental Operations - Multiplication
A-PAK #5, Fundamental Operations - Division
A-PAK #6, Special Products
A-PAK #7, Division of Polynomials
A-PAK #8, Linear Equations
A-PAK-#9, Linear Inequalities
A-PAK #10, Fractions
A-PAK #11, Graphing Numbers, Sets, and Ordered Pairs
A-PAK #12, Factoring
A-PAK #13, More Factoring
A-PAK #14, Quadratic Equations
A-PAK #14A, Applying the Quadratic Formula
(An Optional A-PAK for Enrichment)
A-PAK #15, Systems of Equations
A-PAK #16, Graphing Equations
A-PAK #17, Roots
A-PAK #18, Word Problems

Basically, each of the A-PAKS, with the exception of 14A, an optional topic, contains:

1. Statement of Rationale.
2. Performance Objectives.
3. Performance Activities, with an original treatise dealing with the mathematical concept embodied.
4. Performance Test.
5. Performance Test Answer Key.

The A-PAKS were typed in single-space form in compliance with College of Marin conservation policy. The originals were sent to the duplication center at the college on June 19, 1975; one hundred copies were prepared. Each page of the finished product was printed on both sides of the paper, again to conform with school conservation policy. In order to minimize cost, the finished product was stapled instead of bound. The printed material was then forwarded to the College of Marin bookstore for distribution to the students enrolled in the Technical Mathematics class.

CHAPTER V

PROCEDURES FOR COLLECTION AND TREATMENT OF DATA

The A-PAK Treatment

Twenty-nine students enrolled in the Technical Mathematics course which met for the first class meeting August 20, 1975, at the College of Marin. This class was assigned the title "Experimental Group - Technical Mathematics" since it was scheduled to receive the A-PAK treatment.

A standardized Hundred-Problem Basic Arithmetic Skills Test was administered to the group at the second class meeting.[50] The answer sheets were subsequently scored and the percentile scores determined. The Hundred-Problem Test percentile then served as the basis for selection of a composite "traditional group." This technique was possible because the researcher has historically administered the Hundred-Problem Test as a diagnostic device with traditional Technical Mathematics classes. Thus it was feasible to select a closely paired "traditional score" for each experimental score. For each member of the experimental group there was available a closely matched member for a traditional group.

The experimental group underwent the six weeks of

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50. See Appendix B.

treatment with the prepared materials--the A-PAKS. During this period copious notes were compiled on observations of the experiment by the researcher primarily for future revision of the A-PAKS or in anticipation of any possible residual findings from the experiment.

At the conclusion of the treatment on October 10, 1975, the standardized algebra achievement test was administered to the experimental group.[51] The researcher also had scores for every student who had previously completed the Technical Mathematics class (the traditional group). Thus it was possible for the achievement scores of the experimental group to be statistically compared with the achievement scores of the traditional group.

A subjective questionnaire was also administered to the experimental group in order to tabulate for a like or dislike effect of the A-PAK treatment.[52]

Treating the Hypotheses

In order to statistically test the hypotheses, the data--the Hundred-Problem Test scores and the achievement test scores--were subjected to the following treatment:

1. An analysis of the measures of central location --mean and median were performed.

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[51] See Appendix C.
[52] See Appendix D.

2. A graphic comparison of measures of central location by means of frequency polygons.

3. An analysis of variance utilizing an "F" test of significance was applied to the achievement scores of both experimental and traditional groups. The "F" test was pre-selected for the following reasons:

a. Only one score for each subject was needed for "F" test treatment.

b. The "F" test lends itself to an equal number of subjects in the experimental group and the traditional group.

c. The "F" test is amenable to small size sample experiments--as few as ten to fifteen subjects per group and from two to six groups.[53]

d. Significance is interpretable from the .1 percent to the 20 percent levels, thus allowing for possible wider interpretation.

It was the researcher's pre-stipulation that significance of the experiment would result if the "F" ratio fell at or below the 5 percent (.05) level.

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53. James L. Bruning and B. L. Kintz, Computational Handbook of Statistics.

CHAPTER VI

STATISTICAL PRESENTATION

Hundred-Problem Arithmetic Test

The Hundred-Problem Test was designed for grades seven through twelve inclusive but may be used satisfactorily for high school graduates whether they be in college or employed in industry. The test provides a quick and reliable appraisal of computational skills.

Individual achievement may also be appraised with reference to the national standardization population by locating the percentile rank that corresponds to the student's total raw score on a table provided by the publisher.[54] The percentile indicates the percent of the national population for the given grade that had scores lower than or equal to the student's score.

While the test is not a pure measure of numerical facility, it is constructed so it minimizes other factors like verbal ability.

Table 1 lists each student in the experimental group, his or her Hundred-Problem Test individual raw score, and the individual normed percentile.

Each member of the traditional group was selected from the total population of the years 1970 to 1975.

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[54] For a comprehensive analysis of the test, see the Manual of Directions by Raleigh Schorling, John Clark, and Mary Potter.

Table 1

Hundred-Problem Test Scores and Percentiles of Experimental Group

Student	Raw Score	Normed Percentile
*E-1	22	1
E-2	25	2
E-3	25	2
E-4	25	2
*E-5	29	3
E-6	30	4
E-7	33	5
E-8	33	5
E-9	33	5
E-10	33	5
E-11	35	6
E-12	40	9
E-13	43	11
E-14	43	11
E-15	44	12
E-16	45	13
E-17	49	20
E-18	50	23
E-19	59	35
E-20	64	45
E-21	66	48
E-22	70	54
*E-23	71	56
*E-24	71	56
E-25	72	58
E-26	78	70
E-27	88	89
E-28	89	90
E-29	90	92

*There is a slight percentile difference between the experimental and traditional pairings.

The selection of each member was based on his percentile, matching one in the experimental group; thus the groups became statistically comparable. The traditional group is listed in Table 2.

Hundred-Problem Test frequency tallies. The frequency tally for the experimental group (see Table 3) was constructed with a class interval containing increments of three. Raw scores were utilized since the score range possibility extended from 1 through 100--very neatly lending itself to a graphic interpretation. A slightly different tally was performed for the traditional group frequency (Table 4). The entire population for the years 1970-75 was tallied. This technique provided a larger silhouette for comparison.

Sample means and medians. The experimental group sample mean from the Hundred-Problem Test was computed utilizing the individual percentile scores in lieu of computing the sample mean from frequency distributions since there were few scores and relatively small figures were involved.

When $\overline{X}$ = sample mean:

$$\overline{X} = \frac{X}{N}$$

$$\overline{X} = \frac{832}{29}$$

$$\overline{X} = 28.689$$

$$\overline{X} = 29.7$$

Table 2

Hundred-Problem Test Scores and Percentiles of Traditional Group

Student	Raw Score	Normed Percentile
*T-1	20	1
T-2	24	2
T-3	25	2
T-4	26	2
*T-5	30	4
T-6	31	4
T-7	32	5
T-8	32	5
T-9	32	5
T-10	33	5
T-11	34	6
T-12	40	9
T-13	43	11
T-14	43	11
T-15	44	12
T-16	45	13
T-17	49	20
T-18	51	23
T-19	59	35
T-20	64	45
T-21	66	48
T-22	70	54
*T-23	72	58
*T-24	72	58
T-25	72	58
T-26	78	70
T-27	88	89
T-28	89	90
T-29	90	92

*There is a slight percentile difference between the traditional and experimental pairings.

Table 3

Hundred-Problem Test Frequency Tally of Experimental Group

Raw Score	Midpoint	Frequency	Class Intervals
			98-100
			95- 97
			92- 94
89, 90	90	2	89- 91
88	87	1	86- 88
			83- 85
			80- 82
78	78	1	77- 79
			74- 76
71, 71, 72	72	3	71- 73
70	69	1	68- 70
66	66	1	65- 67
64	63	1	62- 64
59	60	1	59- 61
			56- 58
			53- 55
51	51	1	50- 52
49	48	1	47- 49
44, 45	45	2	44- 46
43	42	2	41- 43
40	39	1	38- 40
35	36	1	35- 37
33	33	4	32- 34
29, 30	30	2	29- 31
			26- 28
25	24	3	23- 25
22	21	1	20- 22
			Below 20

Table 4

Hundred-Problem Test Frequency Tally of Traditional Group (1970-1975)

Raw Score	Midpoint	Frequency	Class Intervals
			99-100
95, 96, 97	96	6	95- 97
92, 94	93	4	92- 94
89	90	2	89- 91
86, 87	87	5	86- 88
83, 84, 85	84	6	83- 85
80, 81, 82	81	4	80- 82
77, 78, 79	78	10	77- 79
74, 75, 76	75	7	74- 76
71, 72, 73	72	6	71- 73
68, 70	69	8	68- 70
65, 66, 67	66	12	65- 67
62, 63, 64	65	12	62- 64
59, 60, 61	60	10	59- 61
56, 57, 58	57	13	56- 58
53, 54, 55	54	13	53- 55
50, 51, 52	51	13	50- 52
47, 48, 49	48	16	47- 49
44, 45, 46	45	14	44- 46
41, 42, 43	42	7	41- 43
38, 39, 40	39	13	38- 40
35, 36, 37	36	7	35- 37
32, 33, 34	33	11	32- 34
29, 30, 31	30	12	29- 31
26, 28	27	4	26- 28
25, 24	24	5	23- 25
20, 21, 22	21	5	20- 22
8, 12, 14, 15, 16, 17, 18, 19	13	14	Below 20

The experimental group sample median for the Hundred-Problem Test was computed utilizing the individual percentile scores. When n is the number of scores in rank order, the measurement with rank $\frac{(n + 1)}{2}$ was the sample median.

$$\frac{(n + 1)}{2} = \text{sample median}$$

$$\frac{(29 + 1)}{2} = \text{15th score, i.e., } \underline{\text{12th percentile}}$$

The traditional group sample mean for the Hundred-Problem Test was computed utilizing the individual percentile scores in lieu of computing the sample mean from frequency distributions in order to obtain a more realistic comparison with the experimental group. The traditional group includes the total population attempting the traditional Technical Mathematics course from Fall 1970 through Spring 1975 at the College of Marin.

When $\overline{X}$ = sample mean:

$$\overline{X} = \frac{x}{N}$$

$$\overline{X} = \frac{7628}{239}$$

$$\overline{X} = 31.916$$

$$\overline{X} = 31.9$$

The traditional group sample median for the Hundred-Problem Test was computed utilizing the individual percentile scores. When n is the number of scores in rank order, the measurement with rank $\frac{(n + 1)}{2}$ was the sample median.

$$\frac{(n + 1)}{2} = \text{sample median}$$

$$\frac{(239 + 1)}{2} = \text{120th score, i.e., 24th percentile}$$

Frequency polygons. Three frequency polygons were plotted, each based on its respective frequency tally. Figure 3 depicts the experiment group; Figure 4 graphs the traditional group. A third frequency polygon was then compiled containing both the experimental and traditional group plots. This third graph (Figure 5) highlights the individuality of the experimental group.

Algebra Placement Test

The Cooperative Mathematics Test (Algebra I) is employed at the College of Marin as a standardized placement test. Every student prior to enrolling in a mathematics course at the college must obtain a satisfactory score on the placement test. In order to enroll in First Year Algebra, the student must obtain a score of sixteen on either Form A or B of the Algebra I test. If the student obtains a score of twenty-six, he may enroll in either Math B (Advanced Algebra) or Math C (Trigonometry). In order to enroll in Calculus, the student must successfully pass both Mathematics B and C or receive a satisfactory score on the Algebra II-Trigonometry placement test.

The researcher has historically administered the

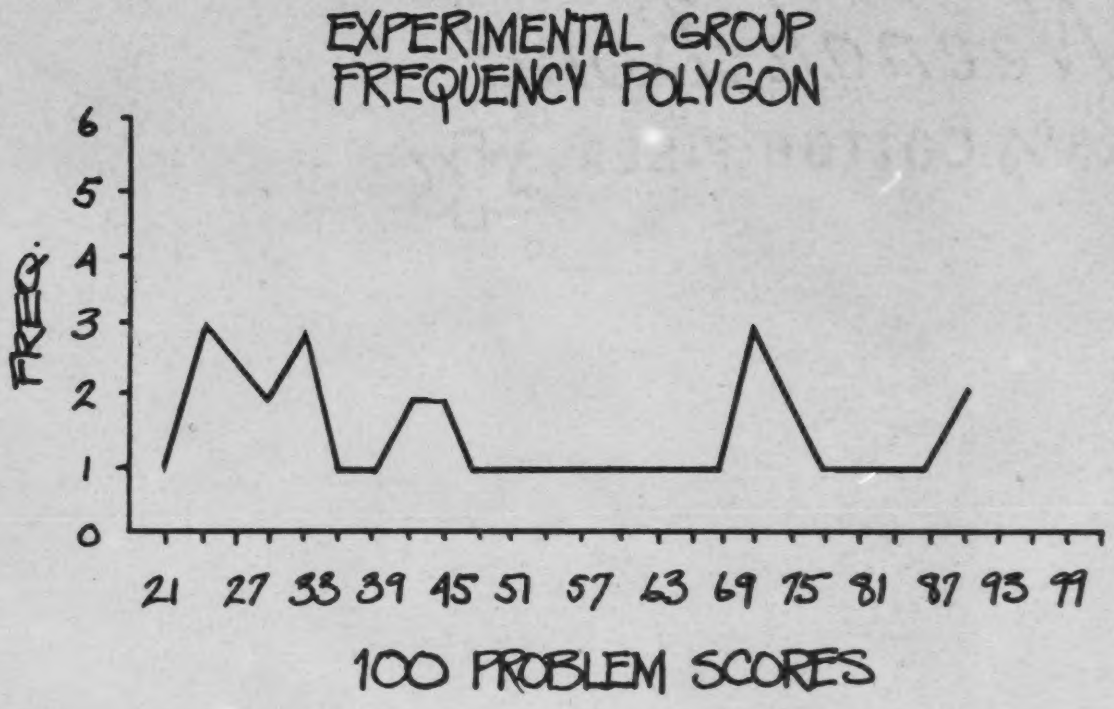
EXPERIMENTAL GROUP
FREQUENCY POLYGON
6
5
4
3
2
1
0
FREQ.
21
27
33
39
45
51
57
63
69
75
81
87
93
99
100 PROBLEM SCORES

Figure 3

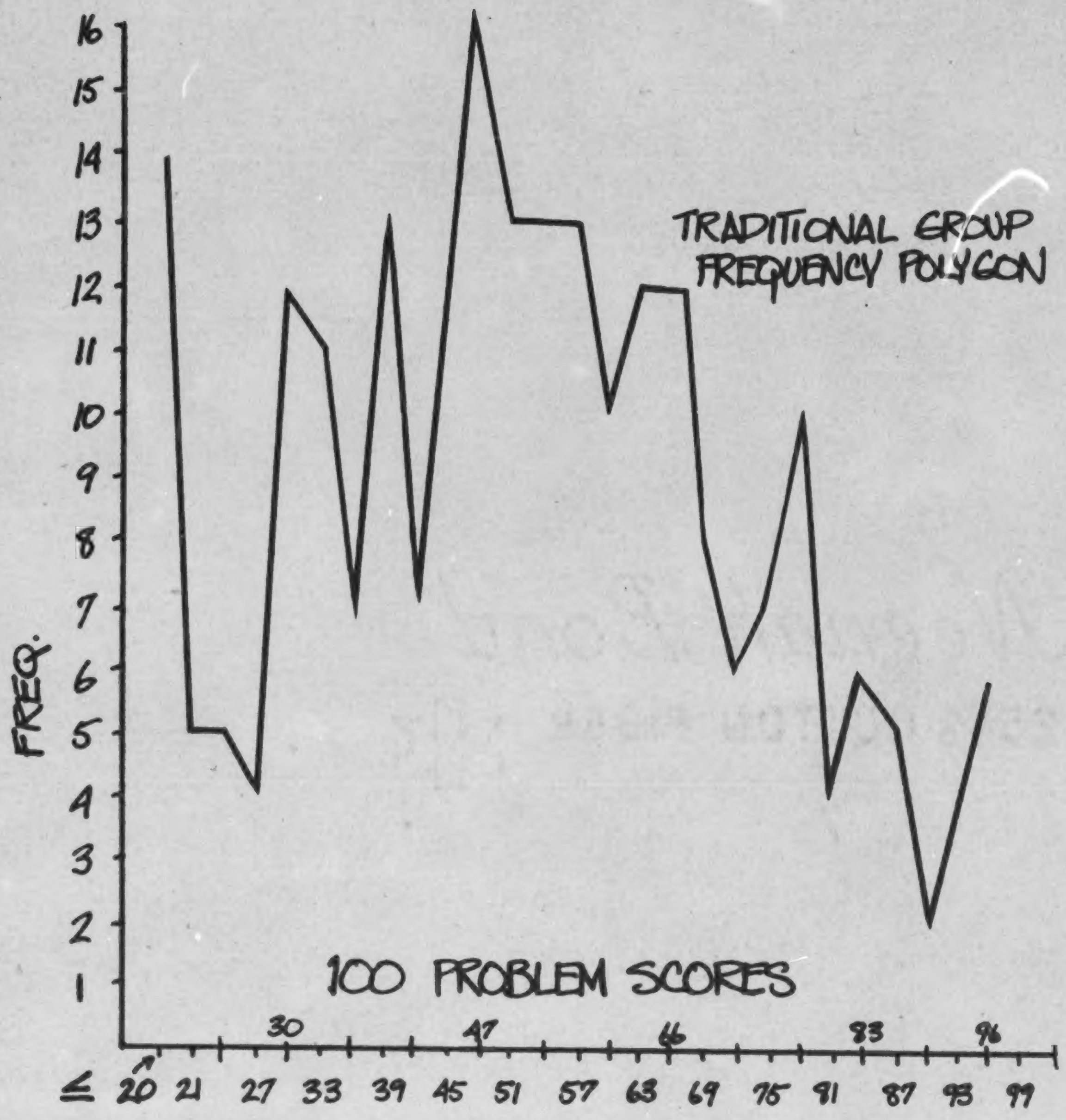
TRADITIONAL GROUP
FREQUENCY POLYGON
FREQ.
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
100 PROBLEM SCORES
30
47
66
83
96
≤
20
21
27
33
39
45
51
57
63
69
75
81
87
93
99

Figure 4

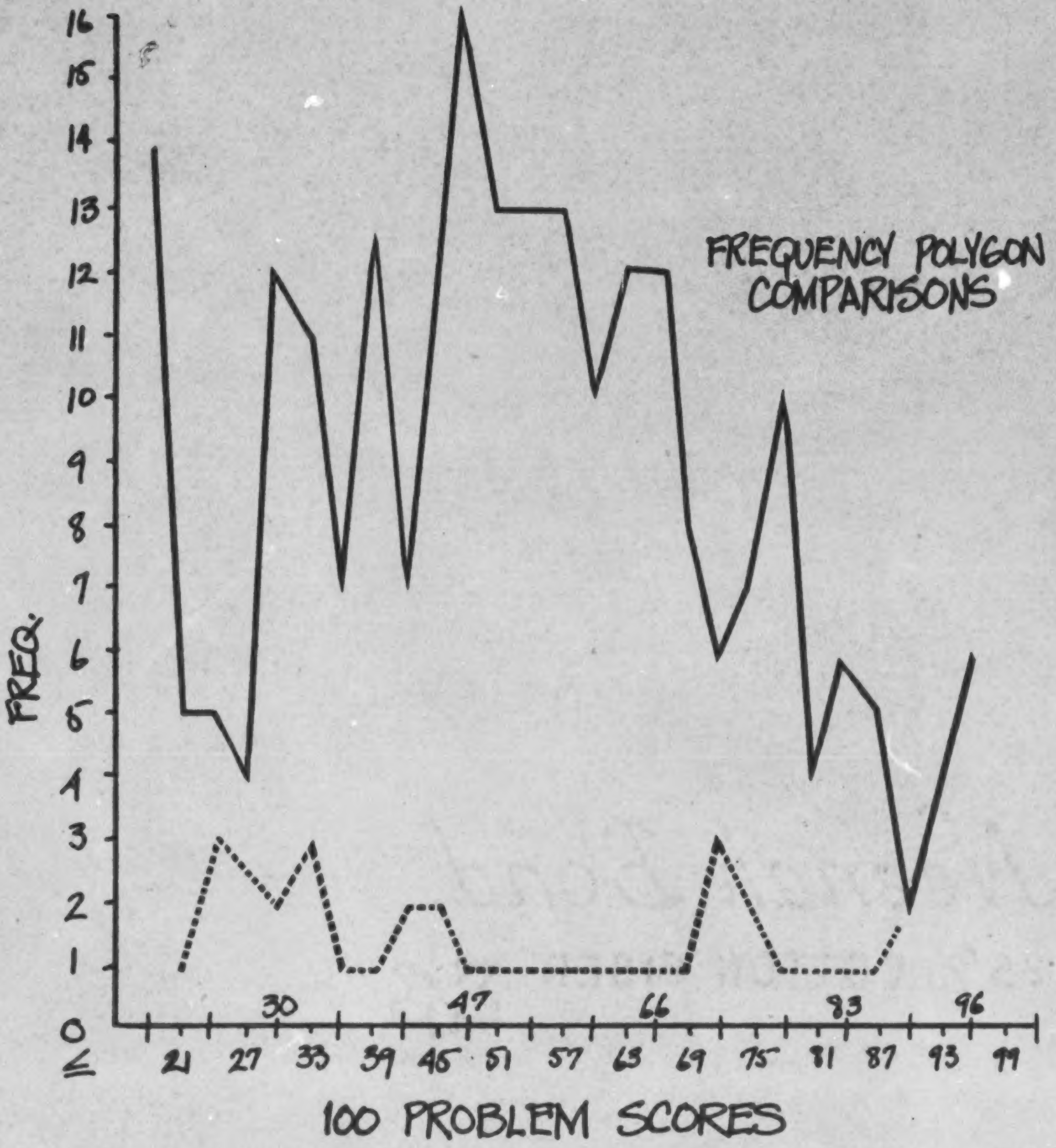
FREQUENCY POLYGON COMPARISONS
FREQ.
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
≤
21
27
33
39
45
51
57
63
69
75
81
87
93
99
30
47
66
83
96
100 PROBLEM SCORES

Figure 5

Algebra I test as a service to his Technical Mathematics classes. Thus scores were available for comparison.

The experimental group raw scores are identified in Table 5; traditional group raw scores in Table 6. The student designation Code E-1, E-2, E-3, T-1, T-2, T-3, etc., is retained from the Hundred-Problem Test assignment.

From the raw scores of both experimental and traditional group frequency tallies, Tables 7 and 8 were tabulated. The class intervals were selected so they would match the intervals chosen for the Hundred-Problem Test tallies. Thus the Algebra Placement Test frequency polygons would hopefully resist any numerical distortion.

Both experimental group and traditional group means and medians were computed utilizing the raw scores in lieu of the frequency distributions since there were few scores and relatively small figures.

For the experimental group sample mean:

$$\overline{X} = \frac{x}{N}$$

$$\overline{X} = \frac{611}{25}$$

$$\overline{X} = 24.4$$

The experimental group sample median was the 13th score, i.e., 24.

For the traditional group sample mean:

Table 5

Algebra Placement Test Scores of Experimental Group

Student	Raw Score#
E-1	9
E-2	30
E-3	14
E-4	17
*E-5	-
E-6	15
E-7	18
E-8	16
E-9	9
*E-10	-
E-11	18
*E-12	-
E-13	16
E-14	27
E-15	16
*E-16	-
E-17	24
E-18	29
E-19	26
E-20	27
E-21	28
E-22	19
E-23	26
E-24	23
E-25	26
E-26	27
E-27	34
E-28	34
E-29	40

*These scores were not available. The students withdrew from the class during the experiment.

#A total score of forty is possible.

Table 6

Algebra Placement Test Scores of Traditional Group

Student	Raw Score#
T-1.................	15
T-2.................	16
T-3.................	14
T-4.................	22
*T-5.................	19
T-6.................	21
T-7.................	8
T-8.................	12
T-9.................	15
*T-10................	10
T-11................	19
*T-12................	16
T-13................	13
T-14................	12
T-15................	11
*T-16................	17
T-17................	20
T-18................	30
T-19................	24
T-20................	32
T-21................	27
T-22................	16
T-23................	25
T-24................	27
T-25................	13
T-26................	31
T-27................	32
T-28................	36
T-29................	37

*These scores were not statistically included. Their experimental group counterparts withdrew from the class.

#A total score of 40 is possible.

Table 7

Algebra Placement Test Frequency Tally of Experimental Group

Raw Score	Midpoint	Frequency	Class Intervals
40	39	1	38-40
	36	0	35-37
34, 34	33	2	32-34
29, 30	30	2	29-31
26, 26, 26, 27,) 27, 27, 28)	27	7	26-28
23, 24	24	2	23-25
17, 18, 18, 19	18	4	17-19
14, 15, 16, 16, 16	15	5	14-16
	12	0	11-13
9, 9	9	2	8-10
			Below 8

Table 8

Algebra Placement Test Frequency Tally of Traditional Group

Raw Score	Midpoint	Frequency	Class Intervals
	39	0	38-40
36, 37	36	2	35-37
32, 32	33	2	32-34
30, 31	30	2	29-31
27, 27	27	2	26-28
24, 25	24	2	23-25
20, 21, 22	21	3	20-22
19	18	1	17-19
14, 15, 15, 16, 16	15	5	14-16
11, 12, 12, 13, 13	12	5	11-13
8	9	1	8-10
			Below 8

$$\overline{X} = \frac{x}{N}$$

$$\overline{X} = \frac{528}{25}$$

$$\overline{X} = 21.1$$

The traditional group sample median was also the 13th score, i.e., 20.

Frequency polygons. In order to obtain a more realistic picture of the Algebra Placement Test scores, three frequency polygons were plotted. Figure 6 depicts the distribution of the experimental group Algebra Placement Test scores based on the experimental group frequency tally. Figure 7 similarly was based on the traditional group frequency tally. For a frequency polygon comparison a composite polygon (Figure 8) was graphed.

F-Test for Significance

Table 9 contains the placement test data summary utilized for the analysis of variance based upon the placement test data.

The following series of calculations were necessary for computing the F-ratio:

Grand total of group sums.

$$611 + 528 = 1139$$

Correction term = grand total square/total of subjects
Correction term = $1139^2/50$
Correction term = $1297321/50 = 25946$

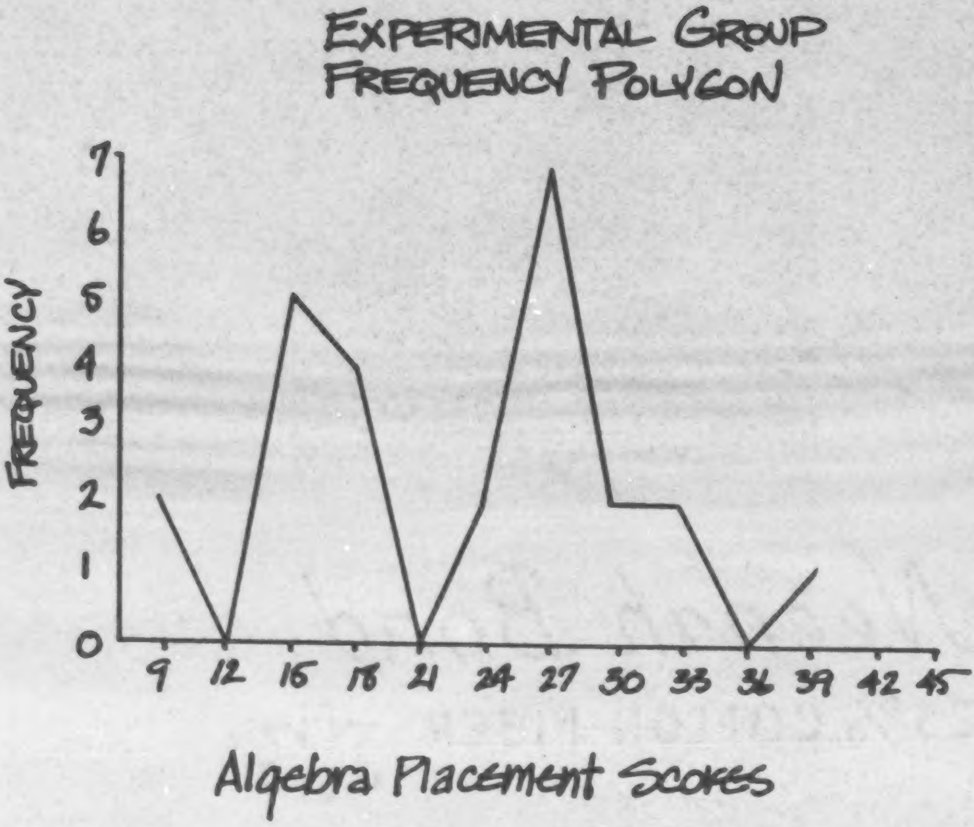
EXPERIMENTAL GROUP
FREQUENCY POLYGON
FREQUENCY
0
1
2
3
4
5
6
7
9
12
15
18
21
24
27
30
33
36
39
42
45
Algebra Placement Scores

Figure 6

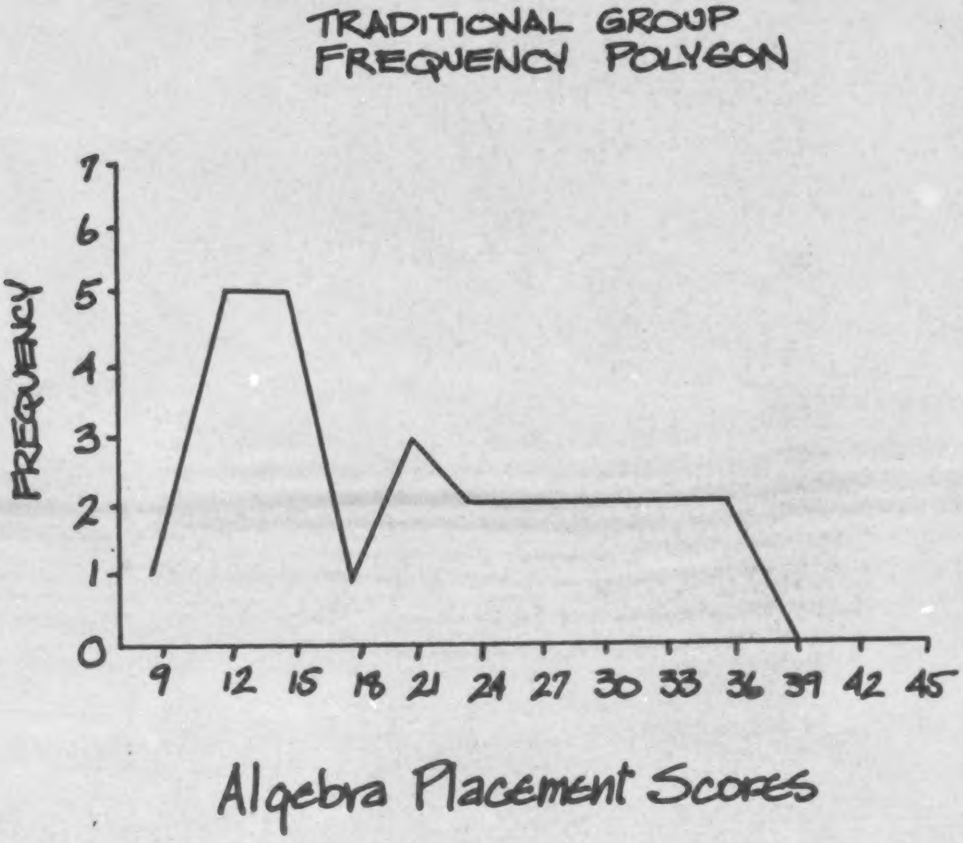
TRADITIONAL GROUP
FREQUENCY POLYGON
FREQUENCY
0
1
2
3
4
5
6
7
9
12
15
18
21
24
27
30
33
36
39
42
45
Algebra Placement Scores

Figure 7

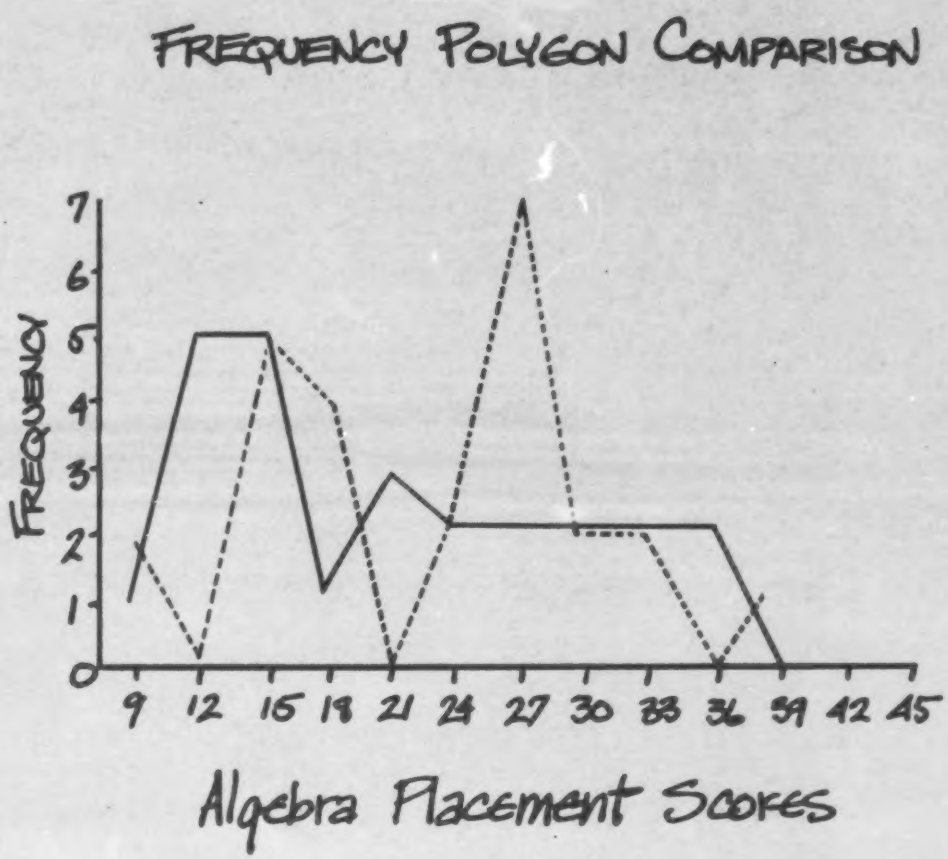
FREQUENCY POLYGON COMPARISON
FREQUENCY
7
6
5
4
3
2
1
0
9
12
15
18
21
24
27
30
33
36
39
42
45
Algebra Placement Scores

Figure 8

Table 9

Data Summary

Experimental Group			Traditional Group		
Student	Score (X)	Score $(X)^2$	Student	Score (X)	Score $(X)^2$
E-1	9	81	T-1	15	225
E-2	30	900	T-2	16	256
E-3	14	196	T-3	14	196
E-4	17	289	T-4	22	484
(E-5)	-	-	(T-5)	-	-
E-6	15	225	T-6	21	441
E-7	18	324	T-7	8	64
E-8	16	256	T-8	12	144
E-9	9	81	T-9	15	225
(E-10)	-	-	(T-10)	-	-
E-11	18	324	T-11	19	361
(E-12)	-	-	(T-12)	-	-
E-13	16	256	T-13	12	144
E-14	27	729	T-14	13	169
E-15	16	256	T-15	11	121
(E-16)	-	-	(T-16)	-	-
E-17	24	576	T-17	20	400
E-18	29	841	T-18	30	900
E-19	26	676	T-19	24	576
E-20	27	729	T-20	32	1024
E-21	28	784	T-21	27	729
E-22	19	361	T-22	16	256
E-23	26	676	T-23	25	625
E-24	23	529	T-24	27	729
E-25	26	676	T-25	13	169
E-26	27	729	T-26	31	961
E-27	34	1156	T-27	32	1024
E-28	34	1156	T-28	36	1296
E-29	40	1600	T-29	37	1369
25	611	14406	25	528	12888

Total $(X)^2 = 14406 + 12888 = 27294$

Sum of squares total (SS_t) = sum of squares - correction term.

$$SS_t = 27294 - 25946$$
$$SS_t = 1348$$

Square of the sum of each group divided by the number of scores in each group.

$$\frac{611^2}{25} + \frac{528^2}{25} =$$

$$\frac{373321}{25} + \frac{278784}{25} =$$

$$14932 + 11151 = 26083$$

Sum of squares between groups (SS_b).

$$SS_b = 26083 - \text{correction term}$$
$$SS_b = 26083 - 25946$$
$$SS_b = 137$$

Sum of squares within groups (SS_w).

$$SS_w = SS_t - SS_b$$
$$SS_w = 1348 - 137$$
$$SS_w = 1211$$

Degree of freedom (df) for components.

df for SS_t : 50 - 1 = 49

df for SS_b : 2 - 1 = 1

df for SS_w : 49 - 1 = 48

Mean squares (ms) for components.

$$ms_t = \frac{SS_t}{df} \quad \text{(not needed)}$$

$$ms_b = \frac{SS_b}{df} = \frac{137}{1} = 137$$

$$ms_w = \frac{SS_w}{df} = \frac{1211}{48} = 25.2$$

$$F = \frac{ms_b}{ms_w} = \frac{137}{25.2}$$

$$F = 5.436$$

Table 10 summarizes the essential data necessary for determining the F-ratio significance.[55]

Table 10

F-Ratio Significance
Summary

Source	SS	df	ms	F-Ratio
Total	1348	49	-	-
Between Groups	137	1	137	5.44
Within Groups	1211	48	25.2	-

An F-ratio of 5.44 is slightly higher than the value required for the 2.5 percent (.025) level which is 5.42 for significance. An F-ratio of 7.31 is needed for significance at the 1 percent (.01) level. It is, therefore, evident that the F-ratio of 5.44 with df equal to 1 and 48 _is not significant_ at the 1 percent (.01) level but _is significant_ at the 2.5 percent (.025) level.

---

[55] Bruning and Kintz, _Op. Cit._, "Appendix D - Per Cent Points in the F Distribution," p. 222.

Student Evaluation Questionnaire

The Student Evaluation Questionnaire was designed to yield a subjective like/dislike effect of the A-PAK treatment. At the conclusion of the experiment the Evaluation Questionnaire was completed by twenty-one members of the class--four students were absent. The questionnaire *was not* signed by the students. Table 11 compiles the responses; however, not every statement was responded to. This accounts for the slight tabulation discrepancy which appears on every statement tally. Furthermore, in order to obtain a definite positive or negative impression, the no-opinion responses were considered neutral and discarded.

Table 11

Student Evaluation Questionnaire
Tally

Statement Number	+	0	-
1...............	14	Not tallied	-
2...............	12	-	-
3...............	15	-	1
4...............	12	-	-
5...............	16	-	-
6...............	14	-	-
7...............	11	-	3
8...............	14	-	-
9...............	11	-	-
10...............	14	-	-
11...............	12	-	-
12...............	14	-	-
13...............	9	-	6
14...............	8	-	4
15...............	16	-	-

CHAPTER VII

ANALYSIS OF DATA

Hundred-Problem Test Comparison

The mean score of the traditional group (Hundred-Problem Test percentiles) was computed as 31.9 while the mean score of the experimental group was computed as 28.7, a difference of 3.2. This is interpreted as not being overly significant. However, the median of the traditional group is the 24th percentile while the experimental group is the 12th percentile or exactly one-half as great as the traditional group.

An examination of both the traditional and experimental group frequency polygons visually identifies the discrepancy in median scores and verifies the similarity in mean scores.

Discounting the large negative outlier (less than the 20th percentile that is expected in a remedial class) of the traditional group, the frequency polygon is fairly symmetric. In fact, the degree of kurtosis is almost normal. However, the experimental group frequency polygon shows a long plateau between scores 48 and 69 or platykurtic.

The experimental group is heavy midway between scores 24 to 48 at the mean or balance point while the traditional group also balances at about score 35. Undoubtedly this was due to the high incidence of the "less than 20 scores" and accounts for the medians spread.

The experimental group was then not as well prepared arithmetically as the traditional Technical Mathematics classes historically. The experimental group mean was slightly less than comparable, but the median score indicated fewer adequately prepared students.

Placement Test Comparison and Analysis

A comparison of the experimental group mean score of 24.4 with the traditional group mean score of 21.1 at first glance does not appear numerically significant. With a sample size of 25 the differential of 3.3 thus tends to under-emphasize a possible statistical significance. However, a comparison of median scores--24 for the experimental group and 20 for the traditional group--somewhat modified this effect.

The experimental group median was 4 higher than the traditional group, thus underscoring a 21 percent differential increase between means and medians. A sample size of 25 with a median differential of 4 is a relatively large range for median disparity. Even more interesting is the fact that while the experimental group was relatively weaker in general mathematical capability it surpassed the traditional group algebraically. Coupled with the condensation factor of the treatment, i.e., six weeks of algebra versus approximately sixteen weeks of a traditional algebra exposure reinforces A-PAK treatment significance.

An analysis of the frequency polygons exposed an interesting silhouette reversal; that is, where on the Hundred-Problem frequency polygon comparison the backdrop is the traditional group polygon, on the placement test frequency polygon the backdrop is the experimental group polygon. This was predicted, of course, by both the means and medians reversal even though the differentials were numerically small.

It is fairly obvious that the traditional group polygon is definitely platykurtic, with one peak at about score 14. However, the experimental polygon, when compared to the traditional polygon, is relatively leptokurtic--more peaked, and it graphically emphasizes the quantity of higher scores obtained by the experimental group.

F-Value

Since the F-value of 5.44 with 1 and 48 degrees of freedom would occur by chance less than 5 percent of the time according to the F-ratio significance table, it is statistically evident that significance of the treatment has been established. (The researcher pre-stipulated the 5 percent (.05) level for significance.)

An F-value of 4.08 is necessary to establish significance at the 5 percent (.05) level of confidence, and an F-value of 5.42 is necessary at the 2.5 percent (.025) level of confidence. Thus a hypothesis determination can be made on the basis of Table 10.

Table 10 F-Value Summary

$5.44 \leq 7.31$ at 1% level
$5.44 \geq 5.42$ at 2.5% level
$5.44 \geq 4.08$ at 5% level

Analysis of Student Evaluation

The number of statements on the Student Evaluation Questionnaire responded to by the subjects was somewhat disappointing although community college students are constantly beseiged by evaluation forms and similar questionnaires. Consequently, saturation might account for the poor numerical response.

Many of the statement responses implied an unconcerned attitude, one of indifference about the treatment. This was reflected by the several zero responses (no opinion) submitted.

Three statements in particular were negatively identified:

No. 7. I did not experience difficulty locating recommended study materials (received three minus marks).

No. 13. The Center for Independent Study was of assistance to me (received six negative marks).

No. 14. The audiovisual, games, special treatments, etc., provided more motivation (received four negative marks).

The concentration of negative marks directed at these three statements strongly indicates some deficiency with respect to the innovative performance activities of the treatment. It would appear likely that the students interested in this type of learning activity experienced frustration at trying to locate the study materials or they received little or no help from the Center for Independent Study. Regardless of the cause, however, it is apparent that a better procedure or system must be devised if the special treatments--games, slides, audiovisual, etc., are to be continued as performance activities.

On the reinforcing side the instrument strongly revealed a student like for the experience. Two statements received sixteen positive marks, the highest number of responses:

No. 5. Learning objectives were clearly identified. I understood what I was supposed to learn.

No. 15. In total, I would rate this an educational experience.

Even more gratifying was the positive mark of fifteen attributed to statement No. 3: I would like to learn more algebra.

The students, by and large, declared their like of the treatment--the format, the student assumption of the responsibility for learning, and the condensation of the material.

CHAPTER VIII

CONCLUSIONS

Summary

The purpose of this Major Applied Research Project (MARP) was twofold. The researcher endeavored to develop a systemized approach and condensation of a traditional sixteen-week algebra course into a six-week (18 hour) presentation. The second phase of the project was the on-site testing of the developed materials with a Technical Mathematics class at the College of Marin.

The MARP was formulated in response to the following student needs:

1. Technical education students are expected to solve relatively complex equations in their respective major areas their first college semester.

2. Many technical education majors are deficient in mathematics, and they particularly experience difficulty with the algebra of numbers. Some students experience extreme frustration and discouragement as a result of their poor algebra capability.

3. Algebra proficiency the first semester would enable technical students to re-enter the formal mathematics sequence in time to complete at least one course in the calculus, thus opening avenues for higher prestige-salary occupations.

A comprehensive review of related literature was conducted prior to the development of the A-PAKS. The review encompassed three distinct areas.

1. Selection and condensation of the most important algebraic concepts.

2. Individualized instruction and the preparation of instructional packets.

3. A search of ERIC for related studies and data bearing on individualized- or systems-type instruction.

The review of the related literature was encouraging. Subsequently, it led to development of the eighteen key algebra packets and also some guidelines for eventual design of the experiment.

An experimental class at the College of Marin was then subjected to the A-PAK treatment. Student scores on an achievement test at the end of the treatment were statistically compared with a traditional composite class formulated on the basis of an arithmetic skills test. An objective analysis of the data enabled the researcher to formulate the following conclusion with respect to his hypotheses.

Determination of Hypothesis

For this experiment the null hypothesis was postulated thusly: Technical Mathematics students subjected to a method of condensed/systemized algebra would score comparable to traditionally treated students on a standardized Algebra Placement Test.

It was further pre-stipulated that significance of the experiment would result providing the F-ratio fell at or below the 5 percent (.05) level of confidence. Under inferential treatment it was statistically determined that the F-ratio did indeed lie below the 5 percent (.05) level; in fact, it resided at the 2.5 percent (.025) level. The F-value was computed as 5.44, and the F-ratio significance table identified an F-ratio of 4.08 as being necessary to establish significance. Since $5.44 \geq 4.08$ at the 5 percent (.05) level of confidence, the researcher rejected the null hypothesis.

The alternate hypothesis was two-tailed and postulated thusly: either the experimental group would score higher or the traditional group would score higher on the standardized Algebra Placement Test.

The researcher elected to accept the alternate hypothesis.

Implication and Significance

One of the most important functions of this MARP was the demonstration that algebra need not be presented to college-level students in a set time frame of a total semester or even with the traditional lecture/textbook pedagogy. The researcher believes this MARP established significance since it was directed toward saving student time through "systemized and condensed" methodology.

Technical students are under mathematical performance pressures, especially during their first semester. This MARP attempted to alleviate some of the pressure by providing basic concepts in approximately one-third of the traditional time--giving algebra capability immediately when it is needed most.

It is interesting to note the incipient community college trend toward "minicourses." Concentrated treatment of subject content is encompassed over a minimal time span in minicourses. This trend is particularly evident in noncredit or adult education classes. A gardening class, for example, might meet for twenty hours, realize the course objectives, and then disband or lead to another aspect of horticulture, i.e., tree pruning or Japanese bonsai.

The researcher sees some educational uses for incorporating this technique, with modification, into the regular credit offerings. Why not, for example, offer mathematics in three topical one-unit courses instead of one three-unit course that treats three or four packed topics. Mathematics, in particular, is amenable to this approach; however, little change in the structure of mathematics curriculum has occurred in the past several years.

Commencing at the high school, the structure is basically:

1. Elementary Algebra.
2. Advanced Algebra.
3. Trigonometry-Analytic Geometry.

Continuing at the four-year college:

4. Calculus (one or two years).
5. Major area of specialization, i.e., Statistics or Logic.

At the community college, mathematics courses include:

1. General Mathematics.
2. Elementary Algebra.
3. Advanced Algebra.
4. Trigonometry.
5. Calculus.

It is, however, at this level that more experimentation with sequence can occur. The researcher believes this MARP breached the aforementioned traditional approach to mathematics by semester branches: algebra, geometry, trigonometry, etc. Perhaps the next module might be:

1. Advanced Algebra.
2. Trigonometry.
3. Geometry--solid.
4. Geometry--plane.
5. Slide Rule or Calculator.
6. Calculus--by topical sections.
7. Probability.
8. Statistics.

9. Mathematics needed by the student at this particular time regardless of sequence.

Emphasis might thus be placed on what the student needs in order to be successful in his selected major field while in school.

An additional educational spin-off of the experiment was underscored by utilizing the "systems approach." The students by and large preferred this type of pedagogy. It provided a means for each student to determine his own pace--amount, performance, and evaluation. The emphasis was shifted from a teacher-centered course to a student-centered one. The teacher's role, while necessary, was basically one of providing expertise--offering but one of several possible performance activities for the student's option. While this type of individualized instruction is rapidly gaining educational acceptance, research in this sphere hopefully will encourage significant experimentation at all educational levels with an ultimate goal of meaningful and realistic education.

This MARP concludes with the researcher's commitment to further develop self-instruction packets for community college mathematics instruction as well as continue to refine and utilize the A-PAKS developed in the execution of this MARP.

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APPENDIX:

A. A-PAKS

B. Hundred-Problem Arith. Test

C. Form A, Algebra I, Cooperative Mathematics Tests

D. Student Evaluation Questionnaire

A-PAKS

A systems approach to Algebra

by Don Greenfield

A-PAK ORIENTATION

Rationale

Education is primarily a process of discovery and encounter for the student; therefore, each student is encouraged to be dependent upon himself for learning. Since the student supplies the motivation, this A-PAK system of "Condensed Algebra Concepts" is designed so that each student can set both his own pace and depth for a personal progression through, and mastery of, the A-PAKS. The treatment of algebra has been condensed to the essentials prerequisite for successful solving of technical problems, both during and following completion of the eighteen PAKS. A total time frame of 6 weeks is allotted for algebra.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Define in writing the components of an A-PAK.

2. Given selected study alternatives, establish and maintain predetermined boundaries or parameters for personal performance.

3. Identify personal goals for attainment by the end of this course.

Performance Activities

1. Attend the orientation meeting on Technical Mathematics - IT 53, Instructor - Don Greenfield.

2. Read "A-PAKS - COMPONENTS" included in this A-PAK.

3. Ask the instructor questions you are concerned about pertaining to the course: grading, requirements, procedures, etc.

4. Scan the list of "Personal Goals" included in this A-PAK and place a check mark beside each goal that you wish to set for yourself.

5. Purchase the textbook recommended for IT 53: Daniel R. Peterson and Gilbert M. Peter, Technical Mathematics, Scott Foresman and Co., Glenview, Illinois, 1974. It can be bought in the college bookstore.

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A-PAK COMPONENTS

A-PAK is an abbreviation for Algebra Packet. An A-PAK is a study guide prepared for the student in a systems approach for individualized instruction in college algebra. Each A-PAK is numbered for both reference and convenience. Each is also titled. The title will identify a key algebraic concept. Each A-PAK will also contain the following components:

Rationale. A brief statement that will inform the student why the topic has been selected or why it is important. Perhaps a thought-provoking equation or puzzle will be presented--hopefully to stimulate interest.

Instructional objectives. Statements of what the learner will be able to do at the end of the A-PAK, each instructional package.

Performance activities. A list of specific activities for the student's consideration. All of these activities are optional. The learner selects only those that he wants or needs to do in order for him to realize the instructional objectives.

Performance test. A brief checkup instrument containing items (questions) or sets of problems designed to indicate to the learner whether or not he has mastered and can apply the algebraic concepts in light of the objectives. This criterion-based test is for the student's information/self-appraisal. It is not for grading purposes. As per school policy, midterm and final examinations will be administered for determining student grades.

Performance test answer key. The answer key enables the learner to quickly score his performance test. If the learner has received a score commensurate with his goals, he should progress to the next A-PAK or enrich his educational experience. If, however, the learner has experienced extreme difficulty solving the test items or used an inordinate amount of time, the learner should recycle back into the system. He should:

a. Review the Instructional Objectives.
b. Repeat or perform more of the Performance Activities.
c. Retake and rescore the Performance Test.
d. Make an appointment with the instructor for exploring remedial possibilities.

List of Goals

By the end of this class I want to be able to:

_______ 1. Understand algebraic concepts.

_______ 2. Be able to solve equations in my major field.

_______ 3. Prepare myself for higher mathematics or a continuation of mathematics courses.

_______ 4. Pass this oourse because it is a requirement for my major.

_______ 5. Increase my computational speed.

_______ 6. Review algebra for other course prerequisites, i.e., chemistry.

_______ 7. Pick up three college units with a reasonable amount of effort.

_______ 8. Get an "A" in math.

_______ 9. Be the "best" student in the class.

_______ 10. Gain some math confidence.

Performance Test

Answer the following questions briefly:

1. What is the purpose of an A-PAK?
2. List the components of an A-PAK and define each.
3. Is it necessary to do every performance activity? Why?
4. How much study is necessary for this class?
5. List the three most important goals that you wish to attain as a result of this course. Add more if you so desire.

Performance Test Answer Key

1. An A-PAK is a study guide in a systems approach toward learning basic algebraic concepts. A-PAK stands for Algebra Packet.

2. The components of an A-PAK are:

   a. An identifying number and title.

   b. Rationale. A statement of purpose or an interesting problem for stimulation.

   c. Instructional objectives. What the student will be able to do after completing the A-PAK.

   d. Performance activities. Things to do that will facilitate learning.

   e. Performance test. Problems that will test knowledge or skills.

   f. Performance test answer key. Will enable the performance test to be scored.

3. No, it is not necessary to do every performance activity. The student should set his own goals and decide for himself the amount of work needed to achieve these goals.

4. Subjective. The student must decide, test, reevaluate.

5. Goals are subjective. Each student must decide for himself. However, the student should be realistic in his choices.

A-PAK #1
OPERATIONAL ORDER AND NUMBER PROPERTIES

Rationale

Pythagorian Theorem is generally stated $a^2 + b^2 = c^2$ utilizing an algebraic symbolism as a language. Just as every language contains rules for application, algebra similarly is treated in accordance with number properties and an order of operations.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Match number properties with their algebriac equivalents.

2. Given a selected series of statements about number properties, identify the true statements.

3. Simplify prescribed algebraic expressions by means of rules and order.

4. Express a personal recognition of why number properties are necessary.

Performance Activities

1. Attend the chalkboard review of Number Properties (a regular class meeting).

2. Read "Operational Order and Real Number Properties" included in this A-PAK.

3. Study pages 2-4 and 86-88 in *Technical Mathematics*.

4. Memorize the Number Properties included in this A-PAK.

5. Practice your skills with the following problem sets: Exercise 1, pp. 4-5, Problems 1 through 38, odd numbered problems only.

6. Read pages 1-22 in *Elementary Concepts of Sets*, Edith J. Woodward, Henry Holt & Company, New York, 1959.

REAL NUMBER PROPERTIES

Just as in the game of baseball, algebraic operations must be performed according to specific rules (or regulations). These basic rules are called properties of numbers and were completely formulated by about the year 1900. Some of the properties may seem very simple--in fact, obvious, but in order to construct a number system it is logical to start with simple building blocks and progress to more complex structures. Note that the number properties are reversible; for example, if $a = b$, then it is also true that $b = a$. Instead of using natural (or whole numbers) numbers to state the properties of numbers, variables (symbols) are substituted in order to emphasize the point that the number properties are true for all numbers.

Field Properties[1]

For all real numbers symbolized by a, b, c, and d, the following real number properties are true:

Closure Property

$a + b =$ real number

$a \cdot b =$ real number

The closure property states simply and precisely that when a real number is added to a real number the solution is a real number. The closure property is also true for multiplication.

Associative Property of Addition

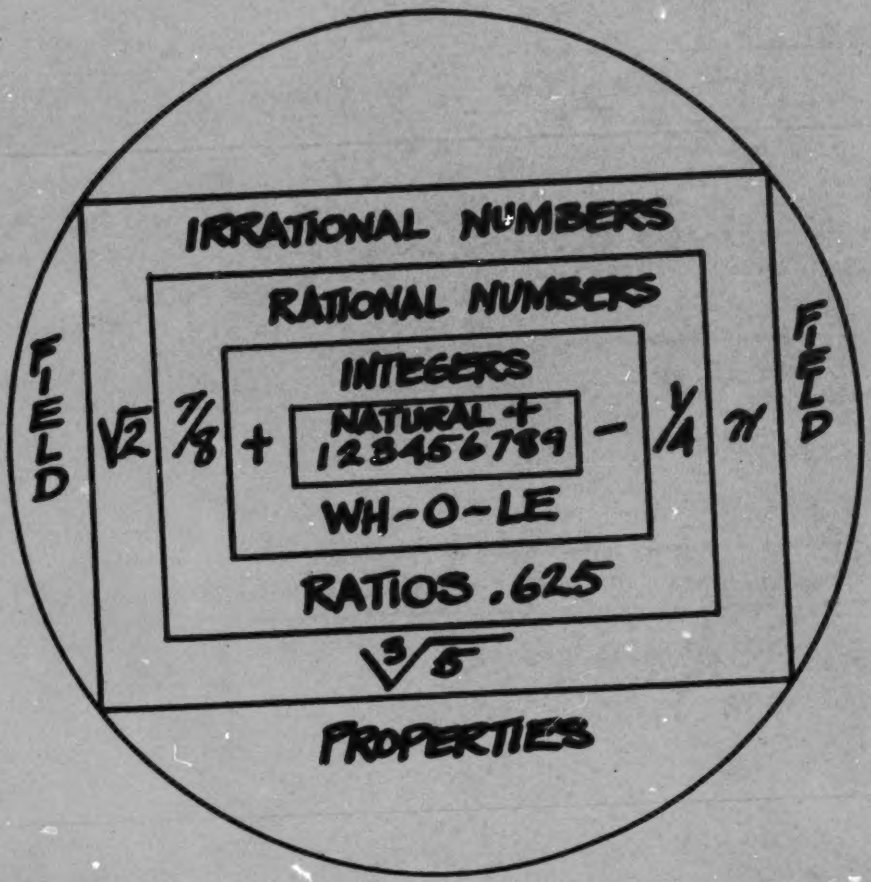
$$(a + b) + c = a + (b + c)$$

The associative property of addition enables addition to be performed on pairs of numbers. We could add a to b and then add the sum to c or we could regroup and add b to c and then add its sum to a. The associative property for addition simultaneously enables parentheses to be cleared.

---

[1] See Diagram

REAL NUMBERS


IRRATIONAL NUMBERS
RATIONAL NUMBERS
INTEGERS
NATURAL +
123456789
WH-O-LE
RATIOS .625
FIELD
FIELD
√2
7/8
+
-
1/4
π
∛5
PROPERTIES

Associative Property of Multiplication

$$(a\cdot b)\cdot c = a\cdot(b\cdot c)$$

Multiplication is also associative.

Commutative Property of Addition

$$a + b = b + a$$

The commutative property of addition allows the addition to be performed by adding the a to the b or the b to the a regardless of order.

Commutative Property of Multiplication

$$a\cdot b = b\cdot a$$

Multiplication is also commutative.

Distributive Property of Multiplication Over Addition

$$a(b + c) = ab + ac$$

The distributive property allows the multiplication before the addition. It also permits the parentheses to be removed. Note that the expression:

$$ab + ac =$$

may be factored--reexpressed as the product of:

$$a(b + c)$$

An expression may be distributed over subtraction:

$$a(b - c) = ab - ac$$

Multiplicative Inverse

$a \cdot \frac{1}{a} = 1$

Every real number except 0[2] has a reciprocal. When the number is multiplied by its reciprocal, the product is equal to 1.

Additive Inverse

$a + (-a) = 0$

The additive inverse states that every number has a unique number that is its opposite, and when they are collected the sum is 0. Note also:

$$a - (+a) = 0$$

Therefore: $a + (-a) = a - (+a)$

Additive Identity

$a + 0 = a$

When 0 is added to any number, the sum is equal to the number. The additive identity enables us to treat 0.

Multiplicative Identity

$a \cdot 1 = a$

The number 1, when multiplied by any number is equal to the number. The multiplicative identity informs us that the number retains its identity. Note also:

$$a = \frac{a}{1}$$

For any number a except zero, the number divided by 1 equals the number.

---

[2] $a \neq 0$

OPERATIONAL ORDER

Signs of inclusion include:

( ) parenthesis, [ ] brackets, { } braces.

They are utilized to indicate a preferred order of mathematical operations. For example:

$(4 + 7) \cdot 6$ indicates that (a) the 4 should be added to the 7 and (b) the sum should be multiplied by 6.

When grouping symbols (the signs of inclusion) are contained within other grouping symbols, the mathematical operations are first performed on the inner pairs as:

$$[(4 + 7) + 3(2 + 4)] = ?$$

$$[11 + 3(6)] = ?$$

$$[11 + 18] = 29$$

If there is uncertainty in solving an expression or equation, or when a specific order is not implied following the removal of signs of inclusion, the preferred operational order is (a) multiplication or (b) division; then (c) addition or (d) subtraction.

NUMBER PROPERTIES MEMORY AID

Closure: $a + b =$ real number

$a \cdot b =$ real number

Associative: $(a + b) + c = a + (b + c)$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Commutative: $a + b = b + a$

$a \cdot b = b \cdot a$

Distributive: $a(b \mp c) = ab \mp ac$

Multiplicative Inverse: $a \cdot \frac{1}{a} = 1$

Additive Inverse: $a + (-a) = 0$

Additive Identity: $a + 0 = a$

Multiplicative Identity: $a \cdot 1 = a$

Performance Test

1. List the following sequence of operations in their correct order:

$$\frac{[2(3 + 4) + 16]}{10} + 3 =$$

2. Illustrate the distributive property over addition.

3. Illustrate the associative property of multiplication.

4. The property $a \cdot b$ = a real number is called ____________.

5. Is it possible to distribute over multiplication as

$$a(b \cdot c) = ab \cdot ac$$

6. $7 + 8 = 8 + 7$ is an example of what number property?

7. Illustrate the additive inverse.

8. Identify the number property $a + o = a$.

9. Is division commutative?

10. What is the purpose of number properties?

Performance Test Answer Key

1. Sequence of operations: (a) $3 + 4 = 7$; (b) $2(12) = 24$; (c) $24 + 16 = 30$; (d) $30/10 = 3$

2. $a(b + c) = ab + ac$

3. $(a \cdot b)\, c = a(b \cdot c)$

4. Closure property.

5. No, multiplication cannot be distributed. Let $a = 2$, $b = 3$, $c = 4$

   $$a(b \cdot c) = a \cdot b \cdot a \cdot c$$

   $$2(3 \cdot 4) = 2 \cdot 3 \cdot 2 \cdot 4$$

   $$24 \neq 48$$

6. Commutative of addition.

7. $a + (-a) = 0$

8. The additive identity.

9. Division is not commutative. $4/2 = 2/4$

   $2 \neq 1/2$

10. Number properties are like rules; they permit only certain operations to be performed.

A-PAK #2
FUNDAMENTAL OPERATIONS - ADDITION AND SUBTRACTION

Rationale

There is only one operation in mathematical calculations--addition. From addition, subtraction is defined. Multiplication is but a fast way to add, and division is defined from it. Addition then is the keystone of mathematical operations.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Given selected problems, perform addition and subtraction algebraically and determine solution sets.
2. Identify from a specified list the correct definitions for new words and terms.
3. Write a definition of absolute value mathematically.
4. Select the number properties that pertain to addition and subtraction from a provided list.
5. Demonstrate how subtraction is accomplished by means of addition.

Performance Activities

1. Attend the chalkboard talk on Addition and Subtraction of Integers and Variables.
2. Read "Addition and Subtraction" included in this A-PAK.
3. Study pages 7-10 and 91-93 in Technical Mathematics.
4. Perform the following practice exercises: Exercise 2, page 11, problems 1 through 40 (odd numbered problems only); Exercise 19, page 94, problems 1 through 50 (odd numbered problems only).
5. Read "Absolute Value" in Elementary Concepts of Sets, Edith J. Woodward, Henry Holt & Co., New York, 1959.

ADDITION AND SUBTRACTION

Addition

When integers are added, note that they signify a quantity of some type of real item. For example:

5 steel parts + 7 steel parts = 12 steel parts

When variables are added, think of a symbol being substituted for the items. Instead of steel parts let the symbol a be the substitute.

$$5a + 7a = 12a$$

The number portion of each term is called the numerical coefficient. The numbers are collected, and only if every symbol to the right of the numerical coefficient is identical can each term be added.

$$5a + 4b + 2a = 7a + 4b$$

Note that it is not possible to add 7a to 4b. This operation can only be algebraically symbolized since a and b are different items.

Defining Absolute Value

If +6 is compared to -6, it is obvious that (other than the signs) the integers are identical. In order to communicate this sameness, the concept of absolute value is beneficial. Absolute value is symbolized by two vertical lines:

$|x|$ = the absolute value of x

If x is a positive number ($x = +2$), then:

$$|x| = x$$

$$|+2| = +2$$

If x is a negative number ($x = -2$), then:

$$|x| = -x$$

$$|-2| = -(-2)$$

$$|-2| = +2$$

If x is 0, then:

$$|x| = 0$$

If x is not equal to 0 ($x \neq 0$), then:

$$|x| = |-x|$$

$$|2| = |-2| \quad \text{or} \quad 2 = 2$$

In the following problem, the first term does not show a numerical coefficient; therefore it is only one item and is added as follows:

$$ax^2 + 2ax^2 + 4ax^2 = 7ax^2$$

Some of the terms in an equation may be negative. In order to add both positive and negative terms, the following rule can be followed:

> If the signs are identical, add the absolute values and use the same sign.

Example 1: $+4 \ (+) \ +5 =$

$|+4| + |+5| =$

$4 + 5 = +9$

Example 2: $-4 \ (+) \ -5 =$

$|-4| \ (+) \ |-5| =$

$4 \ (+) \ 5 = -9$

If the signs are unlike, subtract the lessor from the greater absolute value and use the sign of the greater.

Example 3: $+4 \ (+) \ -5 =$

$|+4|$ subtract $|-5| =$

4 subtract 5 = -1 (use sign of greater)

Example: $-4 \ (+) \ +5 =$

$|-4|$ subtract $|+5| =$

4 subtract 5 = $+1$ (use sign of greater)

Subtraction

Subtraction is defined by means of the Additive Inverse Property.

$$a + (-a) = 0$$

also $\quad a - (+a) = 0$

and $\quad a - (+a) = a + (-a)$

Assume $a = 4$, then:

$$a + (-a) = 0$$

$$4 + (-4) = 0$$

$$4 - 4 \quad = 0$$

And since $a - (+a) = 0$, assume $a = -4$, then:

$$-4 - (-4) = 0$$

-4 and $+4 \quad = 0$

$$-4 + 4 \quad = 0$$

The Additive Inverse Property enables subtraction to be performed by adding the opposite of the subtrahend:

$4x - (+3x) =$

$4x$ say opposite of $(+3x)$ =

or $\quad 4x - 3x = x$

Consider:

$7y + 3x - (-2y) =$

$7y + 3x$ say opposite of $(-2y)$ =

or $\quad 7y + 3x + 2y = 3x + 9y$

Note that when the parentheses are removed, the opposite of the subtrahend is expressed and the problem becomes one of addition.

Consider:

$$3x - (x) =$$

$$3x - x = 2x$$

$$4y - (-2y) =$$

$$4y + 2y = 6y$$

And subtracting vertically:

$$\begin{array}{r} +4y \\ \underline{(-)\ +3y} \end{array} = \begin{array}{r} +4y \\ \underline{-3y} \\ y \end{array}$$

$$\begin{array}{r} 8y^2 \\ \underline{(-)-4y^2} \end{array} = \begin{array}{r} +8y^2 \\ \underline{+4y^2} \\ 12y^2 \end{array}$$

Remember say opposite of change the sign of the subtrahend and then follow the rules for addition.

Performance Test

1. Define the absolute value of x when:
   (a) x is a positive number.
   (b) x is a negative number.
   (c) x is zero.

2. Identify which of the following number properties permit the indicated operations (associative), (commutative), (distributive), (additive inverse), (additive identity):
   (a) $2 + 3 = 3 + 2 = 5$
   (b) $(2c + 2a) + 4c =$
   $(2c + 4c) + 2a = 6c + 2a$
   (c) $5a - (-3a) = 5a + 3a = 8a$
   (d) $2(3x - 5) = 6x - 10$
   (e) $4x + 3x + 0 = 7x$

3. Prove that $23y - (-2y) = 25y$ by identifying a number property that "subtracts" by "adding"!

4. Match the algebraic term to its appropriate definition:

Term	Definitions
numerical coefficient of 2x	$\emptyset$
binomial	$2a + 2$
monomial	2
- (-2)	a
variable	4
null set	$a^2 + 2a + 2$

Perform the indicated operations:

5. $a - 2b + (-a) - (2b) =$

6. From $5x^2 - 6x + 8$ subtract $3x^2 - 2x + 4$

7. If $a = 2$, $b = 3$, $c = 4$, $d = 1$, find
   $2(a + b) + 3(c + d) - (a + b) + 3 =$

8. Add $(2a - b + c)$; $(a - b + c)$; $(3a + b - 2c)$.

9. $95 - (-54) + (-32 + 20) - (-4) =$

10. Add $2a^2 + a - b$ to the sum of $3a^2 + 4a + 2b$ and $a^2 + a - b + 4$.

Performance Test Answer Key

1. (a) $|x| = +x$ when x is positive.
   (b) $|x| = -(x)$ when x is negative (assume $x = -4$)
   $|-4| = -(-4) = 4$
   (c) $|x| = 0$ when x is zero

2. (a) commutative.
   (b) associative.
   (c) additive inverse.
   (d) distributive.
   (e) additive identity.

3. $23y - (-2y) = 25y$ because of the additive inverse
   $x + (-x) = 0$ or $x - (+x) = 0$.

   Thus $23y - (-2y) =$ can be rewritten as
   $23y + 2y = 25y$

4. Numerical coefficient of $2x = 2$
   Binomial $= 2a + 2$
   Monomial $=$ either 2, a, 4
   $-(-2) = 2$
   Variable $= a$
   Null set $= \emptyset$ or the empty set $\{\}$

5. $-4b$

6. $2x^2 - 4x + 4$

7. 17

8. $6a - b$

9. 4

10. $6a^2 + 6a + 4$

A-PAK #3
EXPONENTS

Rationale

Exponents and their fundamental rules are necessary for understanding and applying the base 10 number system. The rules of exponents also permit the definition of mathematical operations when identical bases are treated. A great order and preciseness is obvious as the rules of exponents are developed, and it is evident to some students that there is more to mathematics than mechanical operations. Look for this more than approach.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Identify bases and exponents in given equations or expressions.

2. Carry out mathematical operations--multiplication, division, etc.--by means of the rules of exponents.

3. Given the rules of exponents, write them symbolically.

4. Write a given number as the sum of its constituents and reassemble numbers when given their constituents.

5. Express a recognition of the order and preciseness of higher mathematics.

Performance Activities

1. Read "Exponents" included in this A-PAK.

2. Attend the chalkboard talk on exponents.

3. Memorize "The Rules of Exponents" included in this A-PAK.

4. "Play" the aLPHA GAME, using the rules of exponents. Score your aLPHA GAME with the aLPHA GAME score sheet.

5. Study pages 38-44 in Technical Mathematics.

6. Practice your skills with the following problem sets: Exercise 7, pages 42-43, problems 1 through 55 (odd numbers only).

EXPONENTS

Defining Exponents

Exponents are an abbreviated way of designating repeated multiplication. They may be specific--that is, a set number --or they may be a variable. If the multiplicative factor is variable, it is represented by a symbol that is generally an alphabetical letter. Some examples of bases and exponents are:

X^4 has a base X, exponent 4

12^4 has a base 12, exponent 4

4^X has a base 4, exponent X

M^X has a base M, exponent X

3 has a base 3, exponent 1

The last example (base 3) does not show an exponent. It is understood that any base illustrated without showing an exponent has an exponent of 1. This greatly facilitates the printing of mathematical data since most of the numbers that we deal with daily have an exponential value of 1.

The base always represents a number while the exponent indicates how many times the base is taken as a factor.

7^5 means $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$

X^4 means $X \cdot X \cdot X \cdot X$

X^2 is commonly referred to as X square while X^3 is frequently called X cube. When the exponent is greater than 3, the expression is often designated a power. Thus X^4 is read "X to the fourth power"; similarly, 12^5 is read "twelve to the fifth power." X^n would be read "X to the nth power."

Fundamental Rules of Exponents

The rules of exponents state very precisely how mathematical operations, such as multiplication and division, are carried out when we are dealing with a given base.

Consider: $4^2 . 4^3 = (4.4)(4.4.4)$

$= 4.4.4.4.4$

$= 4^5$

A simplified means of performing this operation--multiplication of like bases--is to add the exponents. This can be precisely stated as RULE I of exponents.

$$X^m \cdot X^n = X^{m+n}$$

Remember, RULE I permits two powers having the same base to be multiplied by simply adding the exponents.

* * *

Consider:

$$\begin{aligned}(4^2)^3 &= (4^2)(4^2)(4^2)\\ &= (4\cdot4)(4\cdot4)(4\cdot4)\\ &= 4\cdot4\cdot4\cdot4\cdot4\cdot4\\ &= 4^6\end{aligned}$$

A simplified means of performing this operation--raising a power to a power: expansion--is to multiply the exponents. We can state this rule mathematically as RULE II.

$$(X^m)^n = X^{mn}$$

Remember, RULE II permits the finding of a power of a base possessing a power by multiplying the exponents.

* * *

Consider:

$$\begin{aligned}\frac{5^5}{5^2} = \frac{5\cdot5\cdot5\cdot5\cdot5}{5\cdot5} &= \frac{5}{5}\cdot\frac{5}{5}\cdot5\cdot5\cdot5\\ &= 1\cdot1\cdot5\cdot5\cdot5\\ &= 5\cdot5\cdot5\\ &= 5^3\end{aligned}$$

A simplified method of accomplishing this operation--division--is to subtract the exponent in the denominator from the exponent in the numerator, or:

$$\frac{5^5}{5^2} = 5^{5-2} = 5^3$$

Mathematically stated, this operation is RULE III.

$$\frac{X^m}{X^n} = X^{m-n}$$

RULE III permits the finding of a quotient of two powers of the same base by taking the difference between the exponents.

* * *

Consider: $\frac{5}{5} = 1 \qquad \frac{M}{M} = 1 \qquad \frac{6^2}{6^2} = 1$

That is, when the numerator and denominator of a fraction are equal, the ratio will reduce to 1.

Thus: $\frac{X^m}{X^m} = 1$

Recall RULE III: $\frac{X^m}{X^n} = X^{m-n}$; and let $m = n$.

Then: $\frac{X^m}{X^m} = X^{m-n} = X^0$

And since: $\frac{X^m}{X^m} = 1$ and $\frac{X^m}{X^m} = X^0$

It follows that X^0 must equal 1. We may state then, any base to the 0 power must equal 1; or RULE IV states $X^0 = 1$ when $X \neq 0$ (zero cannot be a base). This rule permits us to be consistent in our treatment of exponents.

* * *

Consider RULE III: $\frac{X^m}{X^n} = X^{m-n}$

If $m = 0$, then: $\frac{X^0}{X^n} = X^{0-n} = X^{-n}$

And RULE IV: $X^0 = 1$

If $m = 0$, then: $\frac{X^m}{X^n}$ or $\frac{X^0}{X^n} = \frac{1}{X^n}$ or $\frac{1}{X^n}$

We have shown that $\frac{X^0}{X^m} = X^{-n}$ by RULE III.

We have also shown that $\frac{X^0}{X^n} = \frac{1}{X^n}$

Therefore, we have generated RULE V: $X^{-n} = \frac{1}{X^n}$

This rule of exponents encompasses operations involving negative exponents. Remember then when an exponent crosses from the numerator to the denominator or vice versa, the sign of the exponent changes.

$$3^{-2} = \frac{1}{3^2} \qquad \frac{1}{2^2} = 2^{-2} \qquad 3^2 = \frac{1}{3^{-2}}$$

Consider: $25^{\frac{1}{2}}$

This tells us that the base 25 should be multiplied by itself one-half of a time. This is not possible as stated. There is, however, a means of restating the problem by invoking another rule of exponents.

RULE VI*: $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

This rule of exponents enables us to solve problems involving fractional exponents.

Substituting: $25^{\frac{1}{2}} = \sqrt[2]{25^1} = 5$

One more reminder about the rules of exponents. They may be employed in reverse order.

For example: $\sqrt[3]{27^2}$ may equal $27^{\frac{2}{3}}$

---

*RULE VI is also defined as one of the laws of radicals.

Base 10, The Decimal System

We can apply the rules of exponents and construct our number system, base 10.

If the base is 10, then:

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 10 \cdot 10 = 100$$

$$10^3 = 10 \cdot 10 \cdot 10 = 1000$$

If the base is 10 and the exponent is -1, then:

$$10^{-1} = \frac{1}{10^1} = .1$$

If the base is 10 and the exponent is -2, then:

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = .01$$

If the base is 10 and the exponent is -3, then:

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = .001$$

Assembling a base 10 key horizontally, we have:

10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
‖	‖	‖	‖	‖	‖	‖
1000	100	10	1	.1	.01	.001

Numbers

Any number may be reduced to the sum of its constituents using a base 10 key as a guide.

10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
1000	100	10	1	.1	.01	.001

Examine the number 476. It means:

$$\left.\begin{array}{c} 4 \times 10^2 + 7 \times 10^1 + 6 \times 10^0 \\ \text{or} \\ 4 \times 100 + 7 \times 10 + 6 \times 1 \end{array}\right)\quad \begin{array}{l} \text{Reassemble} \\ 400 + 70 + 6 \end{array}$$

Each digit has a positional value as well as a quantitative value.

Examine the number 30.65.

$$\left.\begin{array}{c} 3 \times 10^1 + 0 \times 10^0 + 6 \times 10^{-1} + 5 \times 10^{-2} \\ \text{or} \\ 3 \times 10 + 0 \times 1 + 6 \times .1 + 5 \times .01 \end{array}\right)\quad \begin{array}{l} \text{Reassemble} \\ 30 + 0 + .6 + .05 \\ 30.65 \end{array}$$

THE RULES OF EXPONENTS

RULE I	$x^m \cdot x^n = x^{m+n}$	For multiplication
RULE II	$(x^m)^n = x^{mn}$	For expansion
RULE III	$\frac{x^m}{x^n} = x^{m-n}$	For division
RULE IV	$x^0 = 1$ when $x \neq 0$	For consistency
RULE V	$x^{-n} = \frac{1}{x^n}$	For decimals

You may use this study sheet when playing the aLPHA GAME.

aLPHA GAME

_______1. a^3 = (when a = 4)

_______2. 3^a = (when a = 2)

_______3. a^a = (when a = 3)

_______4. $a^2 \cdot a^3$ = (when a = 1)

_______5. $\frac{a^4}{a^1}$ = (when a = 2)

_______6. a^0 = (when a = 462)

_______7. $\frac{1}{a^{-2}}$ = (when a = 2)

_______8. a^{-2} = (when a = 2)

_______9. $(a^a)^a$ = (when a = 2)

______10. $a^{-1} + a^0 + a^2$ = (when a = 2)

ANSWER KEY FOR aLPHA GAME

1. 64

2. 9

3. 27

4. 1

5. 8

6. 1

7. 4

8. $\frac{1}{4}$

9. 16

10. 5.5 or $5\frac{1}{2}$

Score

10 correct = aLPHA A

8-9 correct = aLPHA B

6-7 correct = aLPHA C

Performance Test

_______1. Identify the base and the exponent of X^m =

_______2. Divide $\frac{36^a}{36^b}$ =

_______3. 10^{-3} =

_______4. $a^2 \cdot a^3$ =

_______5. 75^0 =

_______6. Write as the sum of its constituents the number 782.1

_______7. Write as a number:

$$6 \times 10^2 + 3 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1}$$

Write the rules of exponents (symbolically) using X as the base and m and n as subsequent exponents for:

_______8. Multiplication of like bases.

_______9. Division of like bases.

_______10. Raising a base to a higher power.

_______11. Write a brief paragraph explaining:

a. Why higher math is uninteresting and repetitive

or

b. Why higher math stimulates and challenges.

Performance Test Answer Key

1. Base = X; exponent = m

2. 36^{a-b}

3. .001 or $\frac{1}{1000}$

4. a^5

5. 1

6. $7 \times 10^2 + 8 \times 10^1 + 2 \times 10^0 + 1 \times 10^{-1}$

7. 633.7

8. $x^m \cdot x^n = x^{m+n}$

9. $\frac{x^m}{x^n} = x^{m-n}$

10. $(x^m)^n = x^{mn}$

11. (a) Higher math is generally disinteresting to those students who concentrate on memorizing rules and repeating arithmetical operations.
    (b) Higher math is generally interesting to those students who discover the order and relationship of "abstract symbolism." The mechanics are secondary to the relationships.

A-PAK #4
FUNDAMENTAL OPERATIONS - MULTIPLICATION

Rationale

Multiplying algebraically is slightly more complex because the direction of the number or variable (its sign) must be determined as well as the product.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Indicate the proper sign when given the four multiplication possibilities.

2. Given problems correctly, solve:
   a. Multiplication of integers.
   b. Multiplication of monomials.
   c. Multiplication of polynomials by monomials.

3. Integrate exponential rules while performing given multiplication problems.

4. Perform the multiplication of three or more integers and/or variables and determine the correct sign with confidence and speed.

5. Demonstrate that the sign of a product is correct by means of absolute value.

Performance Activities

1. Attend the Introduction to Multiplication lecture.

2. Read "Multiplication" included in this A-PAK.

3. Do the practice exercises included in this A-PAK and check your solutions with the solution key.

4. Study pages 12-13 and 96-97 in Technical Mathematics.

5. Practice your skill with the following problem sets: page 99, problems 1 through 12 only.

MULTIPLICATION

We learn how to multiply positive numbers by means of basic arithmetic. Recall that multiplication is only a short cut for addition.

$3 \cdot 2$ is equivalent to $2 + 2 + 2$ and the sum is 6

If a and b are positive numbers, then:

$$a \cdot b = |a| \cdot |b|$$

When $a = 2$ and $b = 3$:

$$2 \cdot 3 = |2| \cdot |3| = 6$$

The same logic can be applied when one of the numbers is negative.

$3 \cdot -2$ is equivalent to $-2 \; (+) \; -2 \; (+) \; -2 = -6$

And since multiplication is commutative, then:

$$-2 \cdot 3 = -2 \; (+) \; -2 \; (+) \; -2 = -6$$

If a and b, then, are numbers with opposite signs:

$$a \cdot b = -(|a| \cdot |b|)$$

When $a = +2$ and $b = -3$, then:

$$\begin{aligned} +2 \cdot -3 &= -(|2| \cdot |-3|) \\ &= -(2 \cdot 3) \\ &= -(6) \\ &= -6 \end{aligned}$$

Proving that $-a \cdot -b = +ab$ is a bit more complicat[illegible] However, by means of the number properties it can be accomplished. Try to follow each step carefully in this classic proof:

$1 = 1$	Statement (reflective property)
$1 + (-1) = 1 + (-1)$	additive property of equality
$1 + (-1) = 0$	additive inverse
$-1[1 + (-1)] = 0$	multiplicative identity
$-1(1) + (-1)(-1) = 0$	distributive property

If then: $-1(1) = -1$ (already shown)

And: $-1 + (-1)(-1) = 0$

It must follow that: $(-1)(-1) = +1$

It has now been established that:

$+\cdot+ = +$ and $+\cdot- = -$

$-\cdot- = +$ and $-\cdot+ = -$

PRACTICE EXERCISES WITH THE 1-2-3 METHOD

Multiplying is as easy as 1-2-3.

Step 1: Set the sign.

$$-4a \cdot +5b \cdot -a =$$

Say minus times plus equals minus times a minus equals plus.

$$- \cdot + \cdot - = +$$

Step 2: Multiply numerical coefficients.

$$4 \cdot 5 \cdot 1 = +20$$

Step 3: Multiply like variables by adding exponents.

$$a^1 \cdot a^1 = a^{1+1} = +20a^2$$

$$b^1 = +20a^2b$$

Remember:

Step 1. Set the sign.

Step 2. Multiply numerical coefficients.

Step 3. Multiply variables by adding exponents.

Try this one:

$$+2a^nb^{2y}c \cdot -6a^nb^{2y}c =$$

If you have this answer, you are correct: $-12a^{2n}b^{4y}c^2$

Practice Exercise

1. $b(b^2 + a^4) =$ (distributive property!)

2. $2y(3 + y^4) =$

3. $-2(4 - 4) =$

4. $(-3)(-5) =$

5. $-\frac{1}{2} \cdot -\frac{1}{2} =$

6. $-8ab^3 \cdot -4a^2b^4 \cdot -ab =$

7. $-2(2x - 4y) + 3(2x - 2y) =$

8. $-2x^m(3y^m + x^n) =$

9. $-3 \cdot -3 \cdot +5 \cdot -8 =$

10. $-3x^2y^4z^3(2y^2 - 3x^2y^3z) =$

11. $\begin{array}{r} a^2 + b^2 - c^2 \\ \underline{\qquad\qquad 3a^2} \end{array}$ (multiply vertically)

12. $-\frac{1}{8}x \cdot \frac{1}{4}x^2 =$

13. $(-a)(-a)(-2a)(-3a^2) =$

14. $+7 \cdot -7 =$

15. Find the Area of a rectangle that is x units wide and $x + y - 3$ units long when $A = lw$.

Solution Key for Practice Exercise

1. $b^3 + a^4b$
2. $6y + 2y^5$
3. $-8 + 8 = 0$
4. $+15$
5. $\frac{1}{4}$
6. $-32a^4b^8$
7. $-4x + 8y + 6x - 6y = 2x + 2y$
8. $-6x^my^m - 2x^{2m}$
9. -360
10. $-6x^2y^6z^3 + 9x^4y^7z^4$
11. $3a^4 + 3a^2b^2 - 3a^2c^2$
12. $-\frac{1}{32}x^3$
13. $6a^5$
14. $-?^2$
15. $x^2 + xy - 3x$ sq. units

Performance Test

1. (a) $+\cdot+ =$
   (b) $+\cdot- =$
   (c) $-\cdot- =$
   (d) $-\cdot+ =$

2. By means of absolute value demonstrate that the following is a true statement:

$$+3\cdot-4 = -12$$

3. $2x\cdot-3x =$

4. $5x^2\cdot-3xy\cdot-2xy =$

5. $10x^2(.01x^4 - .1x^2 + .03) =$

6. $-\frac{1}{4}x\cdot-\frac{1}{2}x\cdot-\frac{1}{3}y =$

7. $x^2\cdot x^3\cdot x =$

8. $(7y^{10})(-y^{17}) =$

9. $2x^a(x^{2a} + x^a + x^2) =$

10. $y(3y - 2) - 2y(2 - 4) - 3y^2 =$

Performance Test Answer Key

1. (a) +
   (b) -
   (c) +
   (d) -

2. $+3\cdot -4 = -12$ Since:
$$+3\cdot -4 = -(|+3|\cdot|-4|)$$
$$+3\cdot -4 = -(3\cdot 4)$$
$$+3\cdot -4 = -(12)$$
$$+3\cdot -4 = -12$$

3. $-6x^2$

4. $30x^4y^2$

5. $.1x^6 - x^4 + .3x^2$

6. $-\frac{1}{24}x^2y$

7. x^6

8. $-7y^{27}$

9. $2x^{3a} + 2x^{2a} + 2x^{a+2}$

10. $2y$

A-PAK #5
FUNDAMENTAL OPERATIONS - DIVISION

Rationale

Division as a mathematical operation is a strange technique since it is defined in terms of multiplication! The multiplicative inverse is the special number property that permits simple fractional division; and when the correct rule of exponents is similarly applied, polynomials are easily manipulated.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Demonstrate how the multiplicative inverse permits division of fractions.

2. Write an explanation why division by zero is an invalid operation.

3. Integrate the correct rules of exponents in the division of given problems.

4. Be proficient when dividing polynomials by monomials with both speed and accuracy.

5. Identify the correct sign for the four possibilities of the division of signed numbers.

Performance Activities

1. Attend the chalkboard talk on Reciprocals and the Division of Like Bases.

2. Read "Division of Signed Numbers" included in this A-PAK.

3. Study pages 13, 21-22, and 100 in Technical Mathematics.

4. Memorize the four sign possibilities for division of signs after reading the "Three Steps to Division" included in this A-PAK.

5. Practice your skills with the following problem sets: Exercise 3, page 14, problems 15-18, 23 through 30; Exercise 21, pages 102-103, problems 1 through 12.

DIVISION OF SIGNED NUMBERS

$$a \div b = \frac{a}{b}$$

$$= \frac{a}{b}\cdot 1$$

$$= \frac{a}{b}\cdot\frac{1}{1}$$

$$= \frac{a}{\cancel{b}}\cdot\frac{\frac{1}{b}}{\frac{1}{\cancel{b}}}$$

$$a \div b = a\cdot\frac{1}{b}$$

Note that the numerator is multiplied by the multiplicative inverse of the denominator (its reciprocal); and since 0 does not have a reciprocal, division by 0 is not defined. It is therefore impossible.

When a and b are real numbers with the same sign:

$$\frac{a}{b} = \frac{|a|}{|b|} \qquad \text{i.e.} \quad \frac{+6}{+2} = +3$$

$$\frac{-6}{-2} = +3$$

When a and b are real numbers with opposite signs:

$$\frac{a}{b} = -\frac{|a|}{|b|} \qquad \text{i.e.} \quad \frac{+6}{-2} = -3$$

$$\frac{-6}{+2} = -3$$

Recall the rule of exponents:

$$\frac{x^m}{x^n} = x^{m-n}$$

This rule enables division of variables to be performed:

$$\frac{x^3}{x^2} = x^{3-2} = x^1 = x$$

$$\frac{x^4}{x^8} = x^{4-8} = x^{-4} = \frac{1}{x^4}$$

In the above problem:

$x^{-4} = \frac{1}{x^4}$ is established according to another

rule of exponents:

$$x^{-y} = \frac{1}{x^y}$$

Remember to subtract by adding the inverse if the exponent of the denominator is negative:

$$\frac{x^5}{x^{-3}} = x^{5-(-3)} = x^{5+3} = x^8$$

THREE STEPS TO DIVISION

The division of a polynomial by a monomial is as easy as 1-2-3 providing that you follow these basic steps:

$$\frac{+4a^4b^2c^{-2}}{-2ac^3}$$

Step 1: Divide the signs.

$$\frac{+}{-} = -$$

Step 2: Divide the numerical coefficients.

$$\frac{4}{2} = 2$$

Step 3: Apply the correct rules of exponents.

$$\frac{a^4}{a} = a^{4-1} = a^3$$

$$\frac{b^2}{b^0} = b^2$$

(We assume there is a b^0 in
(the denominator of the prob-
(lem since $b^0 = 1$ and 1 times
(the denominator equals the
(denominator.

$$\frac{c^{-2}}{c^3} = c^{-2-(3)} = c^{-2-3} = c^{-5}$$

Reassemble: $-2a^3b^2c^{-5}$

Remember: $\frac{+}{+} = +$ $\qquad$ $\frac{-}{-} = +$

$\frac{-}{+} = -$ $\qquad$ $\frac{+}{-} = -$

Performance Test

1. (a) $+ \; - \; + \; =$
   (b) $+ \; - \; - \; =$
   (c) $- \; - \; + \; =$
   (d) $- \; - \; - \; =$

2. Explain why this operation is invalid.

   $7 \div 0 = 7$

3. Demonstrate how $\frac{a}{b} \div \frac{c}{d}$ equals $\frac{ad}{bc}$

4. $\frac{1}{4} \div -\frac{1}{2} =$

5. $\frac{5}{8}x \div \frac{1}{2} =$

6. $\left(-20 \div \frac{4}{5}\right) \div \left(-\frac{5}{8}\right) =$

7. $\frac{16x^5 - 4x^4 + 7x^3}{2x^2} =$

8. $\frac{-39x^5 + 26x^4 - 52x^3}{-13x^{-3}} =$

9. $\frac{x^3 + x^2 - x}{-x} =$

10. $(3.5a^8 - .28a^{-7} - .056a^6) \div -7a^5 =$

Performance Test Answer Key

1. (a) +
   (b) -
   (c) -
   (d) +

2. Division is defined as multiplying the numerator by the reciprocal of the denominator. Thus the problem

   $$7 \div 0 = \text{ can be reset } \frac{\frac{7}{1}}{\frac{0}{1}}$$

   And since 0 has no reciprocal, division by 0 is invalid.

3. $$\frac{\frac{a}{b}}{\frac{c}{d}} \cdot 1 = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\cancel{\frac{c}{d}} \cdot \cancel{\frac{d}{c}}}$$

4. $-\frac{1}{2}$

5. $\frac{15}{4}x$

6. $+40$

7. $8x^3 - 2x^2 + \frac{7x}{2}$

8. $3x^8 - 2x^7 + 4x^6$

9. $-x^2 - x + 1$

10. $-.5a^3 + .04a^{-12} + .008a$

A-PAK #6
SPECIAL PRODUCTS

Rationale

Multiplication takes on new significance because it assists in the solving of equations (our ultimate goal). Special products occur repeatedly, and this particular A-PAK is one of the most important since it advocates and encourages some mental gymnastics with binomials.

Instructional Objectives:

After completing this A-PAK, the learner will be able to:

1. Given selected problems, perform the multiplication of monomials, binomials, and complex polynomials.

2. Demonstrate to his own satisfaction the capability of multiplying binomials mentally.

3. Explain why a term disappears with certain binomials when they are multiplied.

4. Multiply selected polynomials vertically.

5. Express an interest in further exploring binomial theory at a future time or in an "extra learning bonus."

Performance Activities

1. Attend the Review of Multiplication lecture that includes binomial products.

2. Read "Patterns of Binomial Products" included in this A-PAK.

3. Study pages 97-99 in Technical Mathematics.

4. Practice your skill with the following problem sets: Exercise 20, p. 99, problems 13 through 38.

5. Play "Concrete Binomial Multiplication." The instructions are included in this A-PAK. The apparatus may be checked out at the Center for Independent Study.

PATTERNS OF BINOMIAL PRODUCTS

With a little practice the product of two binomials can be established without pencil and paper. Here are some patterns that will help you multiply mentally.

Consider: $(a + b)(a + b) = a^2 + 2ab + b^2$

$(a + b)(a + b) = a^2$ (first term)
$a \cdot a$

$(a + b)(a + b) = \quad + b^2$ (last term)
$b \cdot b$

$(a + b)(a + b)$
$b \cdot a = 1ab$
$a \cdot b = (+)\underline{1ab}$
$2ab$ (middle term)

Consider: $(a - b)(a - b) = a^2 - 2ab + b^2$
$a \cdot a = a^2$ (first term)

$(a - b)(a - b)$
$b \cdot b = \quad + b^2$ (last term)

$(a - b)(a - b)$
$-b \cdot a = -1ab$
$a \cdot -b = (+)\ \underline{-1ab}$
$-2ab$ (middle term)

A tricky one: $(a + b)(a - b) = a^2 - b^2$
$a \cdot a = a^2$ (first term)

$(a + b)(a - b)$
$b \cdot b = -b^2$ (last term)

$(a + b)(a - b)$
$b \cdot a = 1ab$
$a \cdot -b = (+)\ \underline{-1ab}$
canceled = no middle term

Practice this one mentally:

$(2a + 3b)(2a + 2b) =$

$4a^2$ (First term)

$6b^2$ (Last term)

$6ab + 4ab = 10ab$ (Middle term)

$= 4a^2 + 10ab + 6b^2$

A perfect square!

$(2a - b)^2 =$

$(2a - b)(2a - b) = 4a^2 - 4ab + b^2$

Since the perfect square of a binomial occurs quite frequently, we can take a short cut with another pattern!

$(2a - b)^2 =$

square the first term $= 4a^2$

square the second term $= +b^2$

multiply the first and second terms $= -2ab$ and double it $= -4ab$

reassemble $= 4a^2 - 4ab + b^2$

Try this one!

$(3a^2 + b^2)^2 =$

square the first term = ___

square the second term = ___

multiply the first and second terms = ___ double it _____

reassemble = $9a^4 + 6a^2b^2 + b^4$

CONCRETE BINOMIAL MULTIPLICATION

The Game*

1. The kit should contain:
   (a) One large square (or several for complex equations).
   (b) Several strips in two colors.
   (c) Several small squares in two colors.

2. The game is essentially one of constructing a graphic model of a given product, either binomial or polynomial. The geometric design is checked against the calculation.

3. The rules are:
   (a) The large square represents x^2, the strips represent x, and the small squares have a constant unit term value of one each.
   (b) If the sign of the term is plus, the major color is used (the color of x^2). When the term sign is negative, the opposite color is used.
   (c) Negative signed terms must be placed of top of positive terms.
   (d) Positives may then be placed on top of negatives if desired.
   (e) The object of the game is to "build" a square or rectangle.

There is no one solution and maybe no solution. This is precisely what makes the game so interesting.

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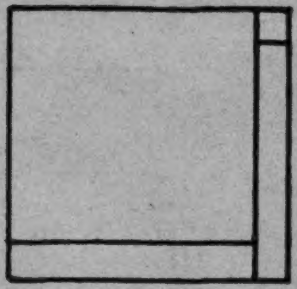
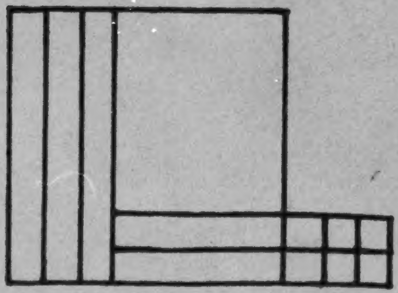
*Credit for this technique should be given to Bob Golton, President, San Francisco Math Teachers Association. I picked up the game idea at The California Mathematics Council Fall Conference, December 6-8, 1974, Asilomar, California.

Examples: $(x + 1)^2 = x^2 + 2x + 1$



All major color +

$(x - 4)(x + 1) = x^2 - 3x - 6$



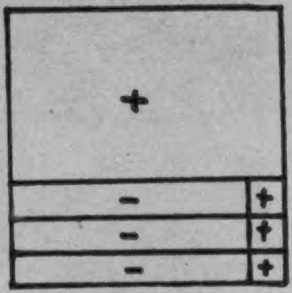
large square +
3 strips - on large
  square vertical
2 + strips horizontal
6 - squares blotting out
  and canceling + overlap

Solve the following problems and check your solutions with the Solution Key. Can you devise any problems? Please diagram the solutions and turn them in.

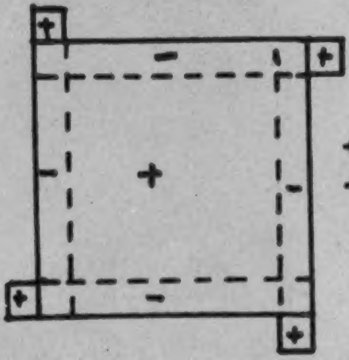
1. $(x - 2)(x - 1) =$
2. $(x - 2)^2 =$
3. $(x + 3)^2 =$
4. $x(x + 2 - 4) =$
5. $(x - 1)^2 =$

Solution Key for Concrete Binomial Multiplication

1. $x^2 - 3x + 3$



2. $x - 4x + 4$



+ on bottom
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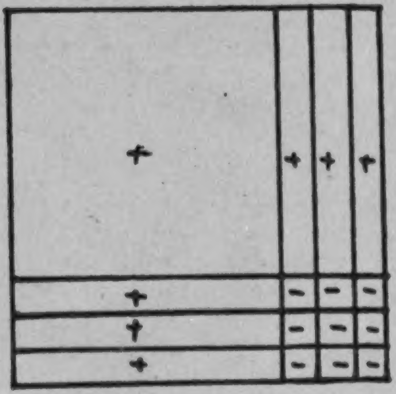
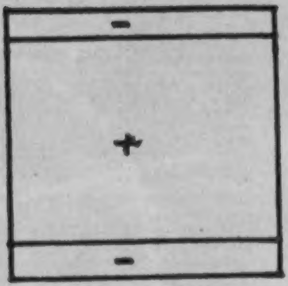
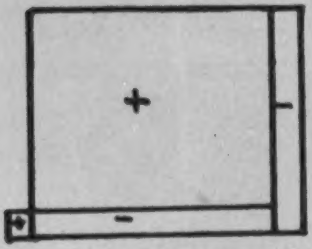
3. $x^2 + 6x - 9$



4. $x^2 - 2x$



5. $x^2 - 2x + 1$



+ on bottom
- on top

Performance Test

1. Explain what happens to the center term when the product of $(x + 3)(x - 3)$ is found.

2. Multiply the following mentally:

   (a) $(x + 2)^2 =$ (c) $(x + y)(x - y) =$

   (b) $(x - 2)^2 =$ (d) $x(x + 3) =$

Find the product of the following:

3. $(3y - 10)^2 =$

4. $(x + 11)(x - 11) =$

5. $(5x - 3)(4x + 7) =$

6. $(c^2 + c + 1)(c + 1) =$ (vertical)

7. $(3a^3 - 2a^2 + a)(2a - 3) =$ (vertical)

8. $(a - 2)(a^2 - 2a + 4) =$ (vertical)

9. $(5x + 2)(7x - 3) =$

10. List three possibilities for further educational experience with special products.

Performance Test Answer Key

1. The center term cancels out when the first terms are identical, the last terms are identical, one binomial is +, and one is -.

2. You should be able to multiply these simple binomials mentally.

   (a) $x^2 + 4x + 4$ (c) $x^2 - y^2$

   (b) $x^2 - 2x + 4$ (d) $x^2 + 3x$ (distributive)

3. $9y^2 - 60y + 100$

4. $x^2 - 121$

5. $20x^2 + 23x - 21$

6. $c^3 + 2c^2 + 2c + 1$

7. $6a^4 - 13a^3 + 8a^2 - 3a$

8. $a^3 - 4a^2 + 8a - 8$

9. $35x^2 - x - 6$

10. (a) Can a pattern be developed for $(a + b)^3$?

    (b) Are there patterns for breaking polynomials into prime factors?

    (c) Is there a three dimensional game for multiplication?

A-PAK #7
DIVISION OF POLYNOMIALS

Rationale

It is relatively simple to determine whether a given integer is a factor of a number. For example, is 5 a factor of 35? Of course, we divide (35 by 5 = 7), and we have determined both 5 and 7 are factors. There are also occasions when we must determine if polynomials are factors of polynomials. In order to verify a factor relationship, so-called "long division of polynomials" can be utilized. Is $(a + b)$ a factor of $(a + b)^3$? Let's see. . .

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Demonstrate that he knows when to employ the long division of polynomials.

2. Specify an appreciation for "long division" as a concluding treatment of mathematical operations.

3. Successfully solve long division of polynomials with given problems.

4. In the event there is a remainder in a given problem, be able to express the result in two ways.

Performance Activities

1. Attend the Development-of-Long-Division chalkboard session.

2. Read "Long Division of Polynomials" included in this A-PAK.

3. Study pages 101-102 in Technical Mathematics.

4. Practice your skill with the following problem sets: Exercise 21, p. 103, problems 13 through 36.

5. Take one of the distribution sheets containing an exposition and practice problems. These will be distributed upon request at the chalkboard session.

LONG DIVISION OF POLYNOMIALS

Sample Problem 1 $(4a + a^2 + 4) \div (a + 2)$

Step 1: Align the variables in descending order of powers.

$$(a^2 + 4a + 4) \div (a + 2)$$

Step 2: Set the problem.

$$a + 2 \overline{\smash{)}\, a^2 + 4a + 4}$$

Step 3: Divide the a^2 by a and multiply the complete divisor back by the quotient.

$$\begin{array}{r l}
 & a \\
a + 2 \big) & \overline{a^2 + 4a + 4} \\
(-) & \underline{a^2 + 2a}
\end{array}$$

Step 4: Subtract. (Remember we subtract by adding the opposite--so change the signs of the subtrahend.)

$$\begin{array}{r l}
 & a \\
a + 2 \big) & \overline{a^2 + 4a + 4} \\
(+) & \underline{-a^2 - 2a \phantom{{}+ 4}} \\
 & \phantom{-a^2 - {}} 2a + 4
\end{array}$$

Step 5: Carry down all terms from the dividend (and the subtrahend if there are any) as above.

Step 6: Divide the first term - the 2a by the a and multiply back.

$$\begin{array}{r l}
 & a \quad + 2 \\
a + 2 \big) & \overline{a^2 + 4a + 4} \\
 & \underline{-a^2 - 2a \phantom{{}+ 4}} \\
 & \phantom{-a^2 - {}} 2a + 4 \\
(-) & \phantom{-a^2 - {}} \underline{2a + 4} \\
 & \phantom{-a^2 - 2a + {}} 0
\end{array}$$

Step 7: Subtract as above.

Solution: $(a^2 + 4a + 4) \div (a + 2) = (a + 2)$

Check: $(a + 2)(a + 2) = (a^2 + 4a + 4)$

Sample Problem 2 $(a^3 + a^4 + 1) \div (-a + a^2 + 1)$

Step 1: Align the variables in descending order of powers.

$$(a^4 + a^3 + 1) \div (a^2 - a + 1)$$

Step 2: Set the problem.

$$a^2 - a + 1 \overline{\big)\, a^4 + a^3 + 1}$$

Step 3: Divide the a^4 by a^2 and multiply the complete divisor back by the quotient.

$$\begin{array}{rl} & a^2 \\ a^2 - a + 1 \big) & \overline{a^4 + a^3 + 1} \\ (-) & \underline{a^4 - a^3 \qquad + a^2} \end{array}$$

(Note a new position
(is established
(since there is no
(a^2 term in the
(dividend.

Step 4: Subtract. (Remember to add the opposite)

$$\begin{array}{rl} & a^2 \\ a^2 - a + 1 \big) & \overline{a^4 + a^3 + 1} \\ (+) & \underline{-a^4 + a^3 \qquad - a^2} \\ & \qquad 2a^3 - a^2 + 1 \end{array}$$

Step 5: Carry down all terms from the dividend and the subtrahend as above.

Step 6: Divide the first term $2a^3$ by the a^2 and multiply back.

$$\begin{array}{rl} & a^2 + 2a \\ a^2 - a + 1 \big) & \overline{a^4 + a^3 + 1} \\ & \underline{-a^4 + a^3 \qquad - a^2} \\ & \qquad 2a^3 - a^2 + 1 \\ (-) & \qquad 2a^3 - 2a^2 \qquad + 2a \\ (+) & \qquad \underline{\;- \qquad + \qquad\qquad -\;} \\ & \qquad\qquad a^2 - 2a + 1 \end{array}$$

Step 7: Subtract and carry down all terms.

Step 8: Divide the a^2 by the a^2 and multiply back.

$$\begin{array}{rl} & a^2 + 2a + 1 \\ a^2 - a + 1 \big) & \overline{a^4 + a^3 + 1} \\ & \underline{} \end{array}$$

Step 9: Subtract.

$$\begin{array}{rl} & a^2 - 2a + 1 \\ (-) & a^2 - a + 1 \\ (+) & \underline{- \quad + \quad -} \\ & \quad -a \end{array} \qquad \text{remainder}$$

Solution: $a^4 + a^3 + 1 \div a^2 - a + 1 = a^2 + 2a + 1$

$$\frac{-a}{a^2 - a + 1} \qquad \text{remainder}$$

Performance Test

1. Identify all problems in the following set that may be solved or proved by means of long division.

   (a) $(a + b)$ is a factor of $a^2 - b^2$?

   (b) Prove that $(a - b)$ is a factor of $(a^3 - b^3)$.

   (c) a is a factor of $a^2 + 2ab$.

   (d) One of the factors of $a^2 - 2ab + b^2$ is $(a - b)$?

2. Identify the mathematical operations necessary for performing long division of polynomials. Is it worthwhile mastering these operations?

3. Express the following remainder in two different ways.

$$\frac{x^2 + 2x - 12}{x - 3} = x + 5 \qquad \text{remainder } 3$$

Solve the following problems by means of long division:

4. $(x^3 - y^3) \div (x - y) =$

5. $(x^3 + y^3) \div (x + y) =$

6. $(b^4 - 16) \div (b - 2) =$

7. $(x^2 - 7x + 12) \div (x - 3) =$

8. $(x^2 - 7x + 12) \div (x - 5) =$

Performance Test Answer Key

1. Long division may prove or solve the following problems: 1a, 1b, 1d.

2. All of the following operations are necessary in order to divide polynomials: addition, subtraction, multiplication, division, and laws of exponents.

   It is worthwhile learning algebraic operations for several reasons, including intellectual exercise, algebra capability for solving technical problems, and learning the language (algebra) of math.

3. $\frac{x^2 + 2x - 12}{x - 3} = (x + 5)(x - 3) + 3$

   or $= x + 5 + \frac{3}{x - 3}$

4. $x^2 + xy + y^2$

5. $x^2 - xy + y^2$

6. $b^3 + 2b^2 + 4b + 8$

7. $x - 4$

8. $(x - 5)(x - 2) + 2$ or $x - 2 + \frac{2}{x - 5}$

A-PAK #8
LINEAR EQUATIONS

Rationale

The pragmatic algebraic goal is to write "open sentences" --equations--and then exercise a capability of solving them. This A-PAK treats polynomial equations in one variable to the first degree.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Given salient information, write open sentences and determine solution sets.

2. Given equations, compose equivalent equations until a solution set is determined.

3. Demonstrate what must be done to specific equations in terms of operations that enables a solution set to be determined.

4. Solve specific equations with a self-designated degree of accuracy in a reasonable amount of time.

5. Identify the axioms of equality that permit given illustrated expressions or substitutions of equality.

Performance Activities

1. Attend the discussion on Equations.

2. Read "Solving Equations" included in this A-PAK.

3. Study pages 119-126 in Technical Mathematics.

4. Practice your skills with the following problem sets: pages 126-127, problems 1 through 127.

5. View "Open Sentences," Eye-Gate 5-6A, at the Center for Independent Study.

6. Write and solve the equations in "Practical Exercises" included in this A-PAK; check your answers with the solution key.

7. Read "Axioms of Equality" included in this A-PAK.

8. View "Linear Equations," Eye-Gate 5-6B, at the Center for Independent Study.

SOLVING EQUATIONS

Equations mathematically describe relationships.

$E = MC^2$ is a scientific relationship where E = energy, M = mass, and C^2 is the speed of light.

$a^2 + b^2 = c^2$ is a geometric relationship that describes the length of the sides of a right angle triangle.

$C = \pi d$ is a relationship between the circumference of a circle, its diameter, and the transcendental number pi (3.1415...).

Equations are solved by substituting a series of equivalent equations until a solution is obtained.

These equivalent equations are composed (or substituted) according to the following axioms. An axiom is a kind of general statement that is accepted as a truth without a formal mathematical proof. Think of an equation as a statement that is in balance; in fact, the equal sign may represent the pivot point of a balance.

$$\underline{3x - 2 = 25}$$

Axiom 1: When the same quantity is added to both sides (members) of an equation, the equation will remain in balance.

$$(+2) + 3x - 2 = 25\ (+2)$$

$$3x = 27$$

Axiom 2: When the same quantity is subtracted from both members of an equation, the equation will remain in balance.

$$3x + 4 = 25$$

$$(-4) + 3x + 4 = 25 - 4$$

$$3x = 21$$

Axiom 3: When both members of an equation are divided by the same quantity, the equation will remain in balance.

$$3x = 21$$

$$\frac{3x}{3} = \frac{21}{3}$$

$$x = 7$$

Axiom 4: When both members of an equation are multiplied by the same quantity, the equation will remain in balance.

$$\frac{4x}{3} = \frac{28}{3}$$

$$(3)\frac{4x}{3} = \frac{28}{3}(3)$$

$$4x = 28$$

It is not necessary to memorize the axioms. Simply ask yourself, "What must be done--addition, subtraction, multiplication, or division, in order to simplify the equation?" Then, perform the operation on both members of the equation. Don't forget to recall the order of operations: "After removing signs of inclusion, multiply or divide; then add or subtract." Next simplify the equation by composing equivalent equations.

PROPERTIES OF EQUALITY

There are three properties of equality.

Reflexive property: $a = a$

Symmetric property: If $a = b$ then $b = a$

Transitive property: If $a = b$ and $b = c$ then $a = c$

The properties of equality hold true for all real numbers a, b, c... These properties assist in the solution of equations.

Consider the following:

$6 + 3 = 9$ then $9 = 6 + 3$ (symmetric)

$2 + 6 = 8$ and $4 + 4 = 8$ then $2 + 6 = 4 + 4$ (transitive)

$9 = 9$ (reflexive)

The properties of equality allow that:

(a) every equation may be reversed.

(b) each real number is equal to itself.

(c) When each of two numbers are equal to a third, then the numbers are all equal.

The properties of equality work in conjunction with the closure properties of addition and multiplication.

PRACTICAL EXERCISES

Solve the following:

1. A 12-foot length of steel must be cut so that one piece is twice as long as the other piece. How long are the pieces?

2. The perimeter of a rectangular patio is 250 feet. The length is 45 feet greater than the width. Find the dimensions.

3. The formula for the circumference of a circle is $C = 2\pi r$. Solve for r.

4. The doctor needed a 25 percent solution of rubbing alcohol but already had 20 quarts of a 15 percent solution on hand. How many quarts of pure alcohol must be added to the 20 quarts?

5. Many machinists like to solve number problems to "keep in shape." Try these!

   (a) One-third of a number is more than 12 by as much as 100 exceeds the number. What is the number?

   (b) When two consecutive even integers are multiplied together and 28 is added, the total is equal to the larger number squared. Determine the numbers.

6. The home economics class devised the following problem to puzzle the teacher. Can you solve it?

   An orange has 29 more calories than a peach and 13 less than a banana. If three oranges have 43 calories less than two bananas and two peaches, how many calories are in an orange?

SOLUTIONS TO PRACTICAL EXERCISES

1. x = length of one piece

$2x$ = length of second piece

$x + 2x = 12$

$x = 4$ and $2x = 8$

2. x = width

$x + 45$ = length

$P = 2L + 2w =$

$250 = 2(x + 45) + 2x$

$40 = x$ and $80 = 24$

3. $C = 2\pi r$

$\frac{C}{2\pi} = r$

4. x = number of quarts of pure alcohol to be added

$20 + x$ = number of resulting quarts

$.15(20) + x = .25(20 + x)$

$x = 2.66$ or $2\frac{2}{3}$ quarts

5. (a) x = the number

$\frac{1}{3}x$ = one-third of the number

$\frac{1}{3}x - 12 = 100 - x$

$84 = x$

(b) x = larger integer

$x - 2$ = the smaller integer

$x(x - 2) + 28 = x^2$ $x = 14,\ x - 2 = 12$

6. x = number of calories in an orange

$x - 29$ = " " " " " peach

$x + 13$ = " " " " " banana

$3x + 43 = 2(x + 13) + 2(x - 29)$

$x = 75$

Performance Test

Write open sentences and solve the following:

1. Together, a lathe and a numerical control milling machine cost $32,000. If the mill cost seven times as much as the lathe, how much did each cost?

2. A 110-foot marine tow cable must be cut so that one piece is 10 feet shorter than twice the length of the other piece. Find the length of the shorter piece.

3. Machine A produces twice as many parts as Machine B. Machine C produces 76 more parts than B. If the total part production is 6,200, how many parts does each machine produce?

4. Identify each operation (add, subtract, etc.) that is necessary in order to solve the following equation:

$$2x + 4 = 28$$

5. Name the axioms of equality that permit the following:

(a) $$\begin{aligned} 3x - 2 &= 25 \\ 3x &= 27 \end{aligned}$$

(b) $$\begin{aligned} 3x + 2 &= 25 \\ 3x &= 23 \end{aligned}$$

(c) $$\begin{aligned} 3x &= 21 \\ x &= 7 \end{aligned}$$

(d) $$\begin{aligned} \frac{4}{3}x &= 28 \\ 4x &= 84 \\ x &= 21 \end{aligned}$$

6. Name the three properties of equality.

Look over the following nine equations. Now specify how much time you are going to need in order to solve them and how many equations you will solve correctly. The purpose is to be realistic in your self-appraisal.

7. $-7 = 2x - 3 - 3x$

8. $2b - [(3b + 4) - 5] - b = 5$

9. $1.5a - (a + 2.4) = 17.4$

10. $0 = -x - 3x - 16$

11. $(|a| - 2) - 3|a| - 16 = 2$

12. $\frac{a}{3} + 5 = -1$

13. $\frac{1}{3}x + 5 = 3$

14. $x^2 + x + 10 = (x + 2)(x - 2)$

15. $8x - 5x = 4800$

Performance Test Answer Key

1. x = lathe
$7x$ = mill
$x + 7x = 32000$
$x = 4000$ (lathe)
$7x = 28000$ (mill)

2. x = one piece
$2x - 10$ = other piece
$x + 2x - 10 = 110$
$x = 40$ (first piece)
$2x - 10 = 70$ (second piece)

3. Machine B = x
Machine A = $2x$
Machine C = $x + 76$
$x + 2x + x + 76 = 6200$
$4x = 6124$
$x = 1531$
$2x = 3062$
$x + 76 = 1607$

4. $2x + 4 = 28$ given

$2x + 4 - 4 = 28 - 4$ add -4

$\frac{2x}{2} = \frac{24}{2}$ divide by 2

$x = 12$

5. (a) Axiom 1 = addition
(b) Axiom 2 = subtraction
(c) Axiom 3 = division
(d) Axiom 4 = multiplication

6. Reflexive, symmetric, and transitive.

7. 4

8. -2

9. 39.6

10. -4

11. -20

12. 12

13. -6

14. -14

15. 1600

An excellent time for all correct is 9 minutes.

A-PAK #9
LINEAR INEQUALITIES

Rationale

We can expand our comprehension and treatment of real numbers by analyzing their order relationship, for order is an integral part of any comparison of numbers. Is it not true that when comparing any two numbers--i.e., a and b--only one of the following is true: a equals b, a is less than b, a is greater than b?

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Specify the three properties of order.

2. Solve selected problems involving integers.

3. Solve given problems involving combined inequalities.

4. Solve specific inequalities by means of equivalent inequalities.

5. Demonstrate an understanding of the order of real numbers by correctly using the addition, subtraction, multiplication, and division principles of order.

Performance Activities

1. Attend the chalkboard talk on Order Relationships.

2. View "Linear Inequalities," Eye-Gate 5-6C, located in the Center for Independent Study.

3. Read "Solving Inequalities" included in this A-PAK.

4. Study "Inequalities" distributed at the chalkboard talk.

5. Practice your skills with the Exercise, Sets I and II, distributed at the chalkboard talk.

6. Read "Equalities and Inequalities," pages 28-33, in Elementary Concepts of Sets.

SOLVING INEQUALITIES

There are three properties of order.

I. The Closure Property--when a real number is added to a real number, the solution is a real number.

II. A Closure Property of Multiplication--when a real number is multiplied by a real number, the product is a real number.

III. The third order property is called the Trichotomy Law. It states that, "If a is a real number, then there are three possibilities of a." They are:

(1) $+a = $ real number
(2) $-(a) = $ real number
(3) $a = 0$

The Trichotomy Law might also be expressed (for all real numbers a and b):

$a > b \qquad a = b \qquad a < b$

There are four symbols that are utilized in stating relationships of order.

$a < b$ read as a is less than b

$a \leq b$ read as a is less than or equal to b

$a > b$ read as a is greater than b

$a \geq b$ read as a is greater than or equal to b

Inequalities, like equalities, are solved by composing equivalent inequalities. However, there is one exception.

Consider the following:

$2x + 1 \geq 5$

$2x + 1 - 1 \geq 5 - 1$ (additive axiom)

$2x \geq 4$

$\frac{2x}{2} \geq \frac{4}{2}$ (division axiom)

$x \geq 2$

Note that any real number greater than 2 satisfies the inequality. Try 3.

$$2x + 1 \geq 5$$

$$2 \cdot 3 + 1 \geq 5$$

$$6 + 1 \geq 5$$

$7 \geq 5$ True.

Now consider:

$$-2x \geq 4$$

$$\frac{-2x}{-2} \geq \frac{4}{-2}$$

$x \geq -2$ Is this true?

Let x = 1 (since 1 is greater than -2):

$$-2x \geq 4$$

$$-2(1) \geq 4$$

$-2 \geq 4$ No.

When each number is multiplied or divided by a negative number, an order reversal occurs. This can be compensated for by reversing the direction of the inequality sign.

$$-2x \geq 4$$

$$\frac{-2x}{-2} \geq \frac{4}{-2}$$

$$x \leq -2$$

$-2x \geq 4$ Check.

$$-2(-3) \geq 4$$

$6 > 4$ True.

Check: let x = -3 since -3 is less than -2.

Performance Test

1. List the three properties of order, either in mathematical symbols or by means of sentences.

2. Mark each of the following T if true and F if false.

   (a) $-5 > 2$

   (b) $-|5| > -6$

   (c) $4z \leq 5z$

   (d) If $a < b$ and $c > 0$, then $ac < bc$

   (e) $|5| > |-5|$

   (f) $-3 \leq 3$

Find a solution for each of the following inequalities by means of equivalent inequalities:

3. $y + 25 > 35$

4. $2x - 3 > 9$

5. $1 - \frac{y}{3} \leq 12$

6. $4x + 3 - 2x < -1$

7. $3(3 + 5) > 9$

8. $2(6x - 8) - 9(x + 4) \geq 2x$

9. $5(a - 4) - (a + 2) \leq 4(a - 4) - 6$

10. $\frac{3}{4}a - \frac{1}{2} \geq \frac{1}{4}$

11. List the principles of order necessary to solve the following inequalities:

    (a) $2x + 3 \leq 7$

    (b) $\frac{3}{4}x > 7$

    (c) $-x < 2$

Solve each of the following combined inequalities:

12. $x + 1 \leq -3$ and $x - 2 \geq -8$

13. $-2 \leq a + 3 < 4$

14. $6b - 3 > 9$ or $6b - 3 < -9$

Performance Test Answer Key

1. Closure of addition, i.e., $a + b =$ a real number.

   Closure of multiplication, i.e., $a \cdot b =$ a real number.

   Trichotomy Law, i.e., if a is a real number, it must be:
   $+a =$ a real number)
   $-(+a) =$ a real number) or $a > b$, $a < b$, $a = b$
   $a =$ zero)

2. (a) F (d) T
   (b) T (e) F
   (c) T (f) T

3. $y > 10$

4. $x > 6$

5. $y \geq -33$

6. $x < -2$

7. $a \geq -2$

8. $x \geq 52$

9. $0 \leq 0$

10. $a \geq 1$

11. (a) subtraction, division.
    (b) multiplication, division.
    (c) multiplication.

12. $-6 \leq x \leq -4$ i.e. $x \geq -6$ and $x \leq -4$

13. $1 > a \geq -5$ i. e. $a < 1$ and $a > -5$

14. $b > 2$ or $b < -1$

A-PAK #10
FRACTIONS

Rationale

Just as the understanding of fractions amplifies the mathematical treatment of "parts of integer wholes," fractional proficiency in the algebra of numbers permits a wider range of polynomial solving. It is a fact that not every mathematical problem possesses neat, orderly integers as equation members or solution sets.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Simplify given algebraic fractions.

2. Specify the rules for multiplying, dividing, adding, and subtracting fractions in a proper mathematical form.

3. Solve specific equations and inequalities that contain fractional numerical coefficients.

4. Solve given equations and inequalities that contain a variable in one or more terms of the denominators.

5. Verify why denominator qualifications are necessary when working with fractional expressions.

Performance Activities

1. Attend the lecture/discussion on Fractions.

2. Read "Fractions" included in this A-PAK.

3. Study pages 14-23 in Technical Mathematics. Work a few problems in each of the accompanying problem sets.

4. Study and solve "Fractional Expressions and Equations" distributed at the lecture/discussion period.

FRACTIONS

Reducing Fractions

Consider: $\frac{ab}{cb}$ This fraction may be rewritten as $\frac{a}{c}\cdot\frac{b}{b}$ or $\frac{a}{c}$ with the only provision that $b \neq 0$ and $c \neq 0$. Thus fractions are reduced or "simplified."

Example 1: (simplify)

$$\frac{x - 5}{x^2 - 25} = \frac{x - 5}{(x + 5)(x - 5)} = \frac{1}{x + 5} \qquad x \neq \pm 5$$

Example 2: (simplify)

$$\frac{21a}{3b} = \frac{7a}{b} \qquad b \neq 0$$

Example 3: (simplify)

$$\frac{2x - 12}{x^2 - 12x + 36} = \frac{2(x - 6)}{(x - 6)(x - 6)} = \frac{2}{x - 6} \qquad x \neq 6$$

A fraction is reduced to lowest terms when the numerator and the denominator do not contain any common factors. In addition, the denominator qualifications are such that the denominator must not be a zero since zero has no reciprocal and the fraction would not be properly defined.

Multiplying Fractions

Consider the following rules for multiplying, dividing, adding, and subtracting fractions.

For all real numbers a b c d.

Rule I: (multiplication) $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$ when $b \neq 0$ and $d \neq 0$

Example: $\frac{(x - 4)}{(x - 3)}\cdot\frac{(x + 6)}{(x + 3)} = \frac{x^2 + 2x - 24}{x^2 - 9}$ when $x \neq \pm 3$

Example: $\frac{-16ab}{4a}\cdot\frac{14a^2y}{14y} =$ (we assume now that (denominators $\neq 0$

$\frac{-4b^2}{1}\cdot\frac{a^2}{1} = -4a^2b^2$

Rule II: (dividing) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

Example: $\frac{a + b}{18} \div \frac{a + b}{3} =$

$$\frac{a + b}{18} \cdot \frac{3}{a + b} = \frac{a + b}{a + b} \cdot \frac{3}{18} = \frac{3}{18} = \frac{1}{6}$$

Rule III: (adding and subtracting)

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \text{ and } \frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

Recall that in order to add or subtract fractions, they must first be expressed with common denominators.

Example: $\frac{3a + 5}{2} + \frac{4a - 1}{3}$ L.C.D. $= 2 \cdot 3 = 6$

$$\frac{3(3a + 5)}{6} + \frac{2(4a - 1)}{6} = \frac{9a + 15 + 8a - 2}{6} = \frac{17a + 13}{6}$$

Example: $\frac{3}{a^3} - \frac{1}{a^2} + \frac{2}{a} =$ L.C.D. $= a^3$

$$\frac{3}{a^3} - \frac{a}{a^3} + \frac{2a^2}{a^3} = \frac{3 - a + 2a^2}{a^3}$$

Example: $b + \frac{2}{b + 4} =$ L.C.D. $= 1(b + 4) = b + 4$

$$\frac{b}{b + 4} + \frac{2}{b + 4} = \frac{b + 2}{b + 4}$$

Example: $\frac{a}{b} + 2 - \frac{b}{a} =$ L.C.D. $= ab$

$$\frac{a^2}{ab} + \frac{2ab}{ab} - \frac{b^2}{ab} = \frac{a^2 + 2ab - b^2}{ab}$$

Fractional Equations and Inequalities

Consider: $\frac{3x}{4} + \frac{8 - 4x}{5} = 3$

Step 1: Find L.C.D. $= 4.5 = 20$

Step 2: Multiply both members of the equation by the L.C.D.

$$20\left(\frac{3x}{4}\right) + 20\left(\frac{8 - 4x}{5}\right) = 20(3)$$

$$15x + 32 - 16x = 60$$

Step 3: Solve. $-x = 28$

$$x = 28$$

Example: $\frac{5}{8}y - \frac{1}{4}y > \frac{3}{2}$

Step 1: L.C.D. $= 8 \cdot 4 \cdot 2 = 64$

Step 2: $64\left(\frac{5}{8}y\right) - 64\left(\frac{1}{4}y\right) > 64\left(\frac{3}{2}\right)$

Step 3: $40y - 16y > 96$

$$24y > 96$$

$$y > \frac{96}{24}$$

$$y > 4$$

Example: $\frac{2}{5}x + \frac{3}{5}x - \frac{2}{7} \leq 1$

Step 1: $5 \cdot 7 = 35$

Step 2: $35\left(\frac{2}{5}x\right) + 35\left(\frac{3}{5}x\right) - 35\left(\frac{2}{7}\right) \leq (1)35$

$$14x + 21x - 10 \leq 35$$

Step 3: $35x \leq 45$

$$x \leq \frac{45}{35}$$

$$x \leq \frac{9}{7}$$

Performance Test

1. List the rules for multiplying, dividing, adding, and subtracting fractions in terms of a, b, c, and d when these variables are not equal to zero.

2. When the following rational expression is reduced as:

$$\frac{5x^2 - 20}{2 + x} = \frac{5(x^2 - 4)}{2 + x} = \frac{5(x + 2)(x - 2)}{2 + x} = 5(x - 2)$$

Explain the denominator qualification $x \neq -2$

Reduce each of the following to reduced form:

3. $\frac{3a + 3}{3a - 6}$
4. $\frac{(x - y)^2}{x^2 - y^2}$
5. $\frac{a^2b}{ab^2}$

Find the product in reduced form:

6. $\frac{1}{a + b} \cdot \frac{1}{a - b} =$
7. $\frac{3x}{4} \cdot \frac{6}{9x} =$
8. $\frac{6}{1 - x^2} \cdot \frac{1 + x}{3} =$

Find the quotient in reduced form:

9. $\frac{3a - 9b}{9} \div (3a - 3b) =$

10. $\frac{8x}{15} \div \frac{5}{4x}$

Express the sum or difference in reduced form:

11. $\frac{1}{2x} + \frac{3}{5x} =$

12. $\frac{a^2}{a - b} - \frac{b^2}{a + b}$

13. $\frac{8}{a - 4} + \frac{4}{a + 2}$

14. $a + \frac{3}{a - 1}$

15. $\frac{a}{b} + 2 + \frac{b}{a}$

Performance Test (continued)

Solve the following equations:

16. $\frac{2}{a} + 3 = 0$

17. $\frac{1}{2x} + \frac{1}{4} = \frac{2}{x}$

18. $\frac{a}{4} + 7 = \frac{a}{2}$

19. $\frac{4}{x} + \frac{3}{3x} = \frac{11}{6}$

20. $4 - \left(\frac{3x - 5}{x}\right) = \frac{10}{x}$

Performance Test Answer Key

1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ $b \neq 0$, $d \neq 0$ (multiplication)

   $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ $b \neq 0$, $c \neq 0$ (division)

   $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$ $b \neq 0$ (addition)

   $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$ $b \neq 0$ (subtraction)

2. In the expression $\frac{5x^2 - 20}{2 + x}$ $x \neq -2$ because:

   If x were -2, the $2 + (-2) = 0$ and the fraction $\frac{5x^2 - 20}{0}$ is then not defined since 0 has no reciprocal.

3. $\frac{3a + 3}{3a - 6} = \frac{3(a + 1)}{3(a - 2)} = \frac{a + 1}{a - 2}$

4. $\frac{(x - y)^2}{x^2 - y^2} = \frac{(x - y)(x - y)}{(x + y)(x - y)} = \frac{x - y}{x + y}$

5. $\frac{a^2b}{ab^2} = \frac{a}{b}$

6. $\frac{1}{a^2 - b^2}$

7. $\frac{1}{2}$

8. $\frac{2}{1 - x}$

9. $\frac{a - 3b}{9(a - b)}$

10. $\frac{32x^3}{75}$

11. $\frac{11}{10x}$

12. $\frac{a^2 + a^2b - ab^2 + b^3}{(a - b)(a + b)}$

13. $\frac{12a}{(a - 4)(a + 2)}$

14. $\frac{a^2 - a + 3}{a - 1}$

15. $\frac{a^2 + 2ab + b^2}{ab}$

16. $a = -\frac{2}{3}$

Performance Test Answer Key (continued)

17. $x = 6$

18. $a = 28$

19. $x = \frac{30}{11}$

20. $x = 5$

A-PAK #11
GRAPHING NUMBERS, SETS, AND ORDERED PAIRS

Rationale

Describe, if you can, a number. Is it a measure of magnitude, quantity, position? Numbers, equations, and inequalities can similarly be described by pictorial means; they can be graphed, utilizing a Cartesian Coordinate system. The graphs themselves may show the solutions to systems of equations or define their parameters.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Graph given numbers, sets, and ordered pairs.

2. Read from supplied graphs the coordinates of numbers, sets, and ordered pairs.

3. Identify the various components of the Cartesian Coordinate system.

4. Graph intervals on the number line when given set data.

5. Specify, by writing, a literal translation of given sets or intervals.

Performance Activities

1. Attend the Graphing Demonstration.

2. Read "Graphs of Real Numbers" included in this A-PAK.

3. Study pages 1-2, 7, 137-138 in *Technical Mathematics*.

4. Practice your skill by graphing the numbers, sets, and ordered pairs included in this A-PAK titled "Pictorial Practice."

5. *Take one* of the distribution packets containing an exposition and practice problems for graphing intervals. These will be distributed upon request at the Graphing Demonstration.

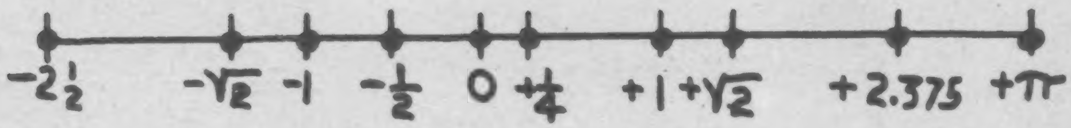
GRAPHS OF REAL NUMBERS

An introduction to graphing is the graphing of real numbers on a number line. Next, sets are treated. Later, the system of assigning numbered pairs in a plane is considered. This latter method is named after its discoverer --Rene Descartes, a French mathematician. "Cartesian Coordinate Graphs" will be constructed; however, it is important that the following elementary graphs be thoroughly understood initially.

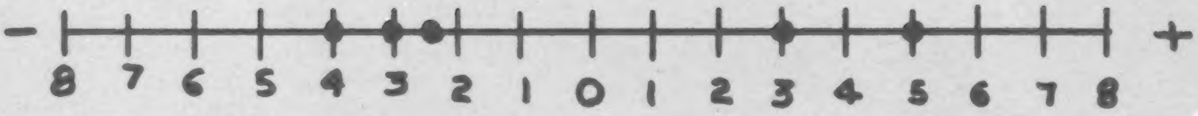
Examine the Number Line A. Note that all of the real numbers can be located as a point on the number line. For convenience, we standardize by dividing the line into uniform divisions and usually represent the positive integers as divisions to the right of zero and the negative integers as divisions to the left of zero. A point assigned to any number is called its graph and is indicated on the number line by a solid dot. A number assigned to a point is called the coordinate of the point.

Locate the graphs of the following real numbers on B: -2.5, -4, +5, -3. When graphing inequalities, a ray is employed. In $x < 5$, an empty dot is used since 5 is excluded; if, however, $x \geq 4$ is graphed, the solid dot is used as it indicates 4 is included. An arrow on the ray indicates it is continuous as illustrated on Graph C.

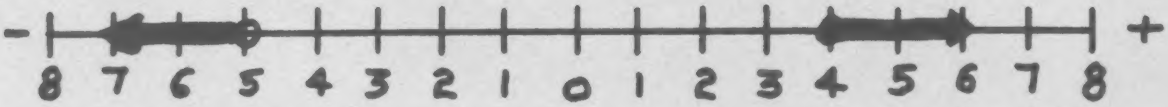
A


−2½
−√2
−1
−½
0
+¼
+1
+√2
+2.375
+π

B


−
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
+

C


−
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
+

Graphing Sets

A set of numbers can be indicated by graphing each of the numbers in the set and following the established methods.

Consider: $Z = \{x \mid -4 \leq x < 7\}$

Read: Z is the set of all x such that x is greater than or equal to 4 and less than 7.

Graphing Intervals

The graph of $D = \{x \mid -3 \leq x \leq 1\}$ includes the point -3 and 1 plus every real number lying between these two points. This kind of a graph is called a closed interval and is graphed according to D.

When graphing a set: $E_a = \{x \mid -2 < x < 2.75\}$ we solve: $x \geq -2$ and $x < 2.75$ and graph as in E_b. There is an overlap and the overlap consists of an open interval--one end open.

Similarly, a half-open interval would be:

$$F = \{x \mid -3 < x \leq 3\}$$

A more complex interval is the graph of:

$$G = \{x \mid |x| \leq 4\}$$

Recall the definition of absolute value:

$$|x| = x, \qquad |x| = 0, \qquad |x| = -(x)$$

We have, then, three possibilities!

$$x > 0, \qquad x = 0, \qquad x < 0$$

If $x > 0$ and $x = 0$, then: $0 \leq x \leq 4$ and this interval is graphed as in G.

If $x < 0$, then $-x > 0$ (its opposite must be greater)

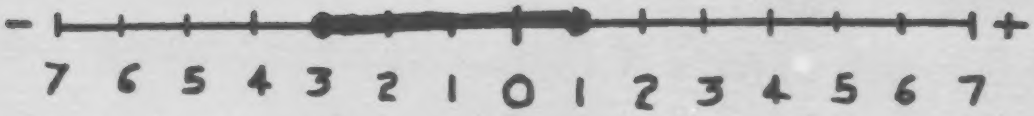
or $\{x \mid 0 < -x \leq 4\}$ or solving for positive x we have:

$\{x \mid 0 > x \geq -4\}$ as graphed in G.

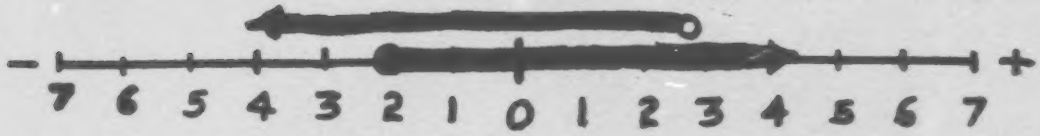
Another possibility is indicated when an or set is graphed:

$H = \{x \mid x < -3$ or $x \geq 2\}$ The or set is graphed as H.

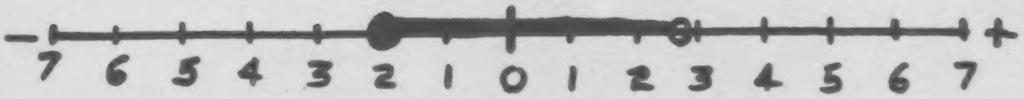
D


−
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
+

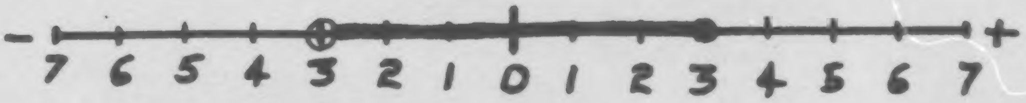
Ea


−
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
+

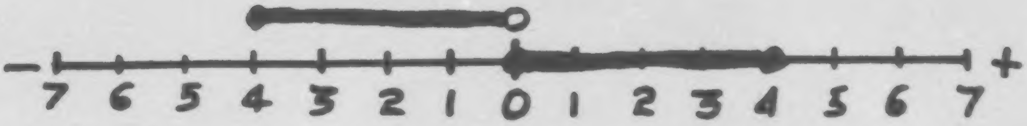
Eb


−
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
+

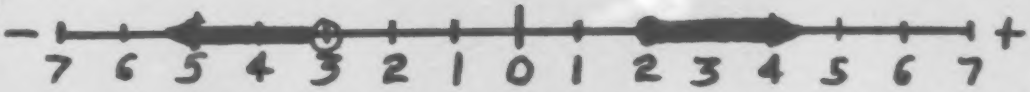
F


−
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
+

G


−
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
+

H

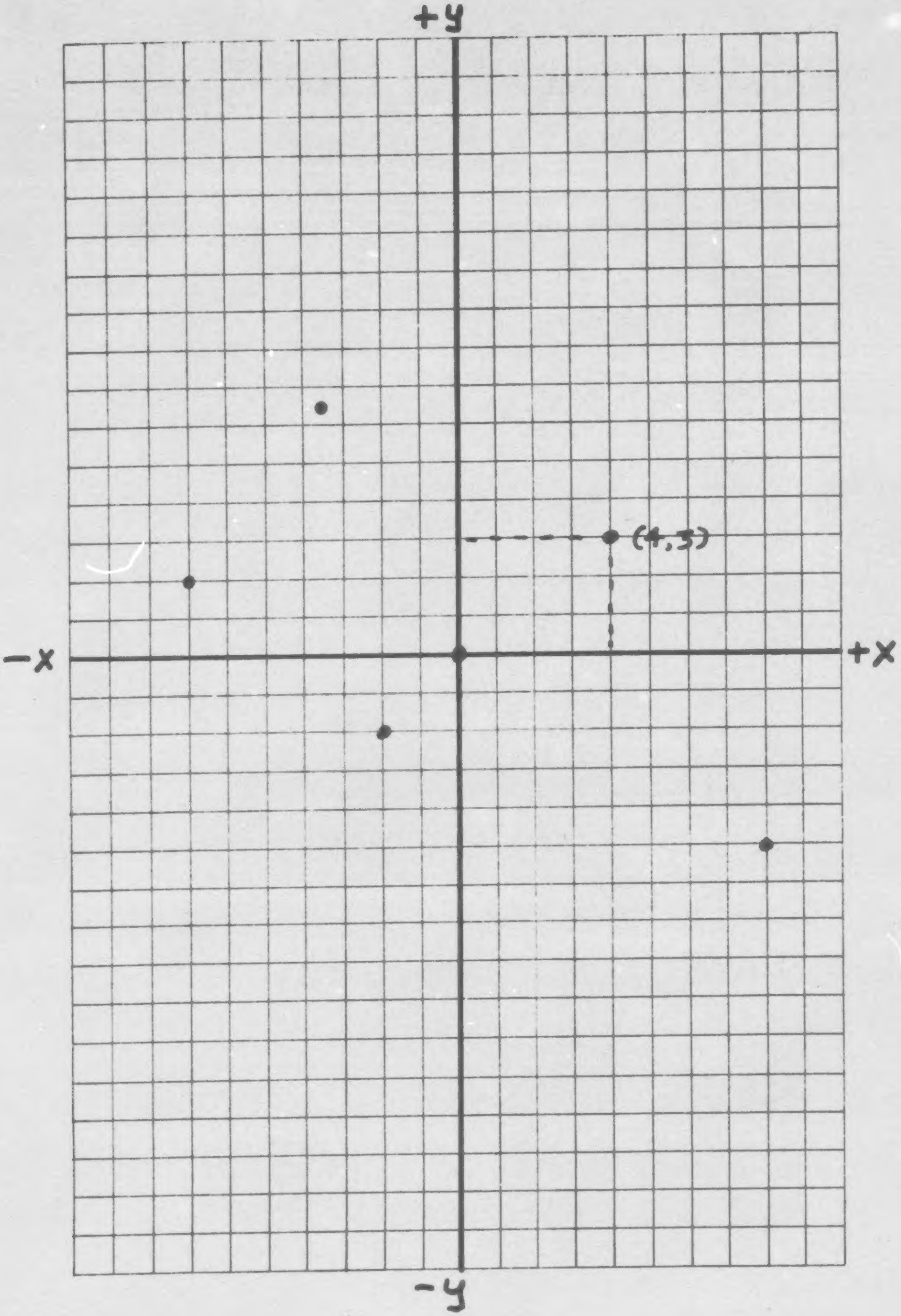

−
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
+

Coordinates in a Plane

Points (ordered pairs) of numbers are graphed on a number plane by means of coordinates. The Cartesian Coordinate Plane is formed by a horizontal number line and a vertical number line. The point of intersection is called the ORIGIN and represents 0. The horizontal line is customarily called the x-axis while the vertical line is termed the y-axis. Positive directions are usually to the right on the x-axis and upward on the y-axis.

In order to locate the graph of an ordered pair (4, 3) we define 4 as the x coordinate and 3 as the y coordinate and plot the point by counting +4 on the x-axis and +3 on the y-axis. At their intersection, we make a dot as shown on the accompanying graph. Where the line from the point meets, the x-axis is called the abscissa of the point; where the line from the point meets the y-axis is called the ordinate of the point. Trace the following points on the graph:

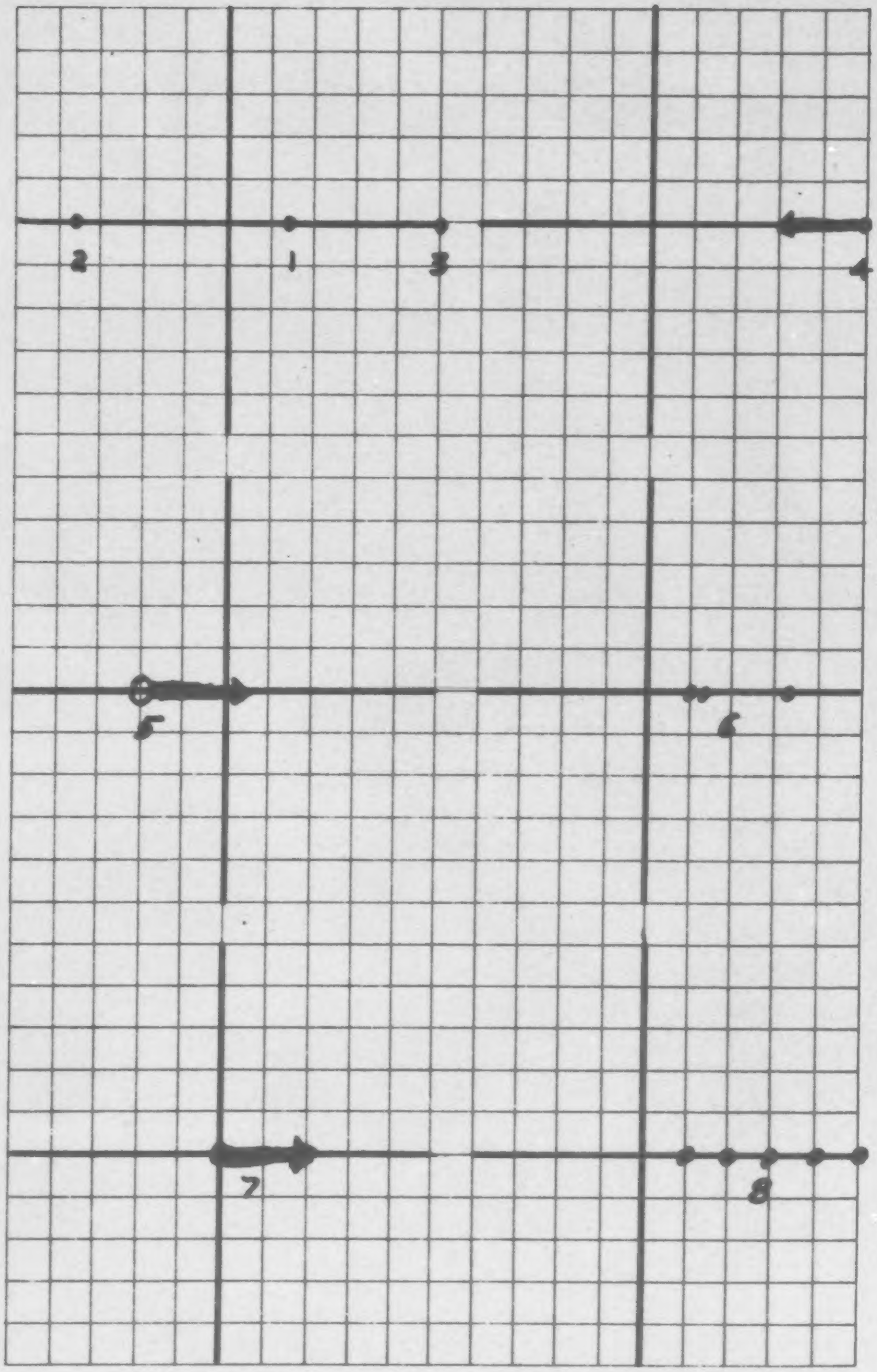
(-7, 2), (-2, -2), (8, -5), (-3.5, 6.5)

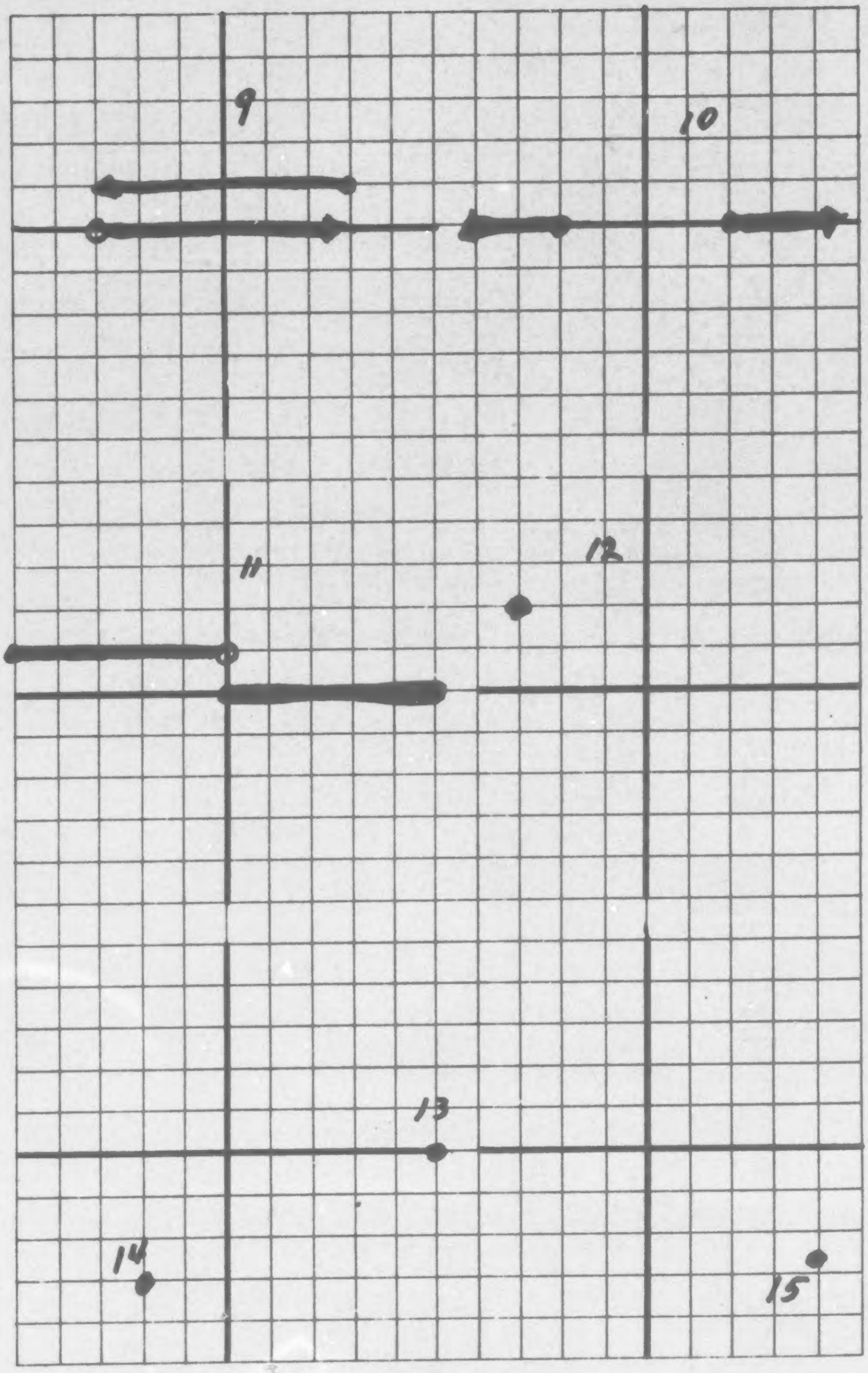

+y
-x
+x
-y
(4,3)

PICTORIAL PRACTICE

Graph the following and check your answers against the answer graphs.

1. $\sqrt{3}$
2. $-3\frac{1}{2}$
3. 5
4. $x \leq 5$
5. $x > -2$
6. $A = \{x \mid x = \pi, \sqrt{2}, 1\}$
7. $B = \{x \mid x \geq 0\}$
8. $C = \{x \mid x = \text{positive integers}\}$
9. $D = \{x \mid -3 < x \leq 3\}$
10. $E = \{x \mid x \leq -2 \text{ or } x \geq 2\}$
11. $F = \{x \mid |x| \leq 5\}$
12. $(-3, 2)$
13. $(5, 0)$
14. $(-2, -3)$
15. $(4, -\frac{5}{2})$





Performance Test

1. Graph (a) -4
   (b) $\frac{9}{4}$
   (c) π

2. Graph (a) $A = \{x \mid 2 \leq x \leq 3\}$

3. Graph $(4, -2)$; $(3, 3)$ $(2, 0)$ $(0, 4)$

4. Graph $B = \{x \mid x =$ all negative integers$\}$

5. Graph $C = \{x \mid x < -1 \text{ or } x > 1\}$

6. Graph $D = \{x \mid x$ is a real number$\}$

7. Read from the attached performance test coordinates the coordinates of each of the points and identify them in correct mathematical form.

8. Align the graph component with its correct definition:

   ordered pair
   abscissa
   ordinate
   origin
   set
   real number

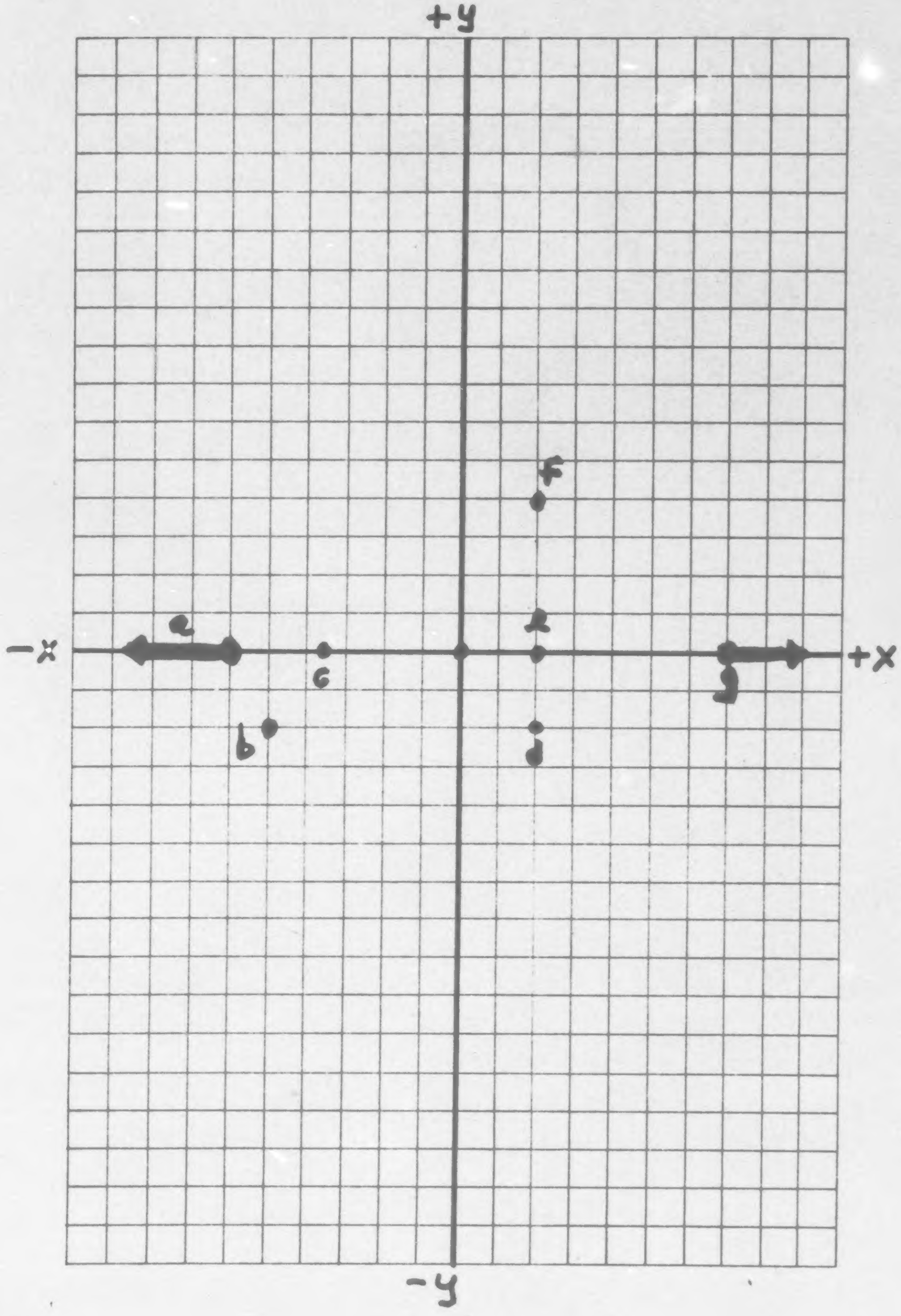
   π
   $(3, -6)$
   x-axis
   y-axis
   0
   $P = \{x \mid x < 2\}$

9. Write out in longhand word for word:

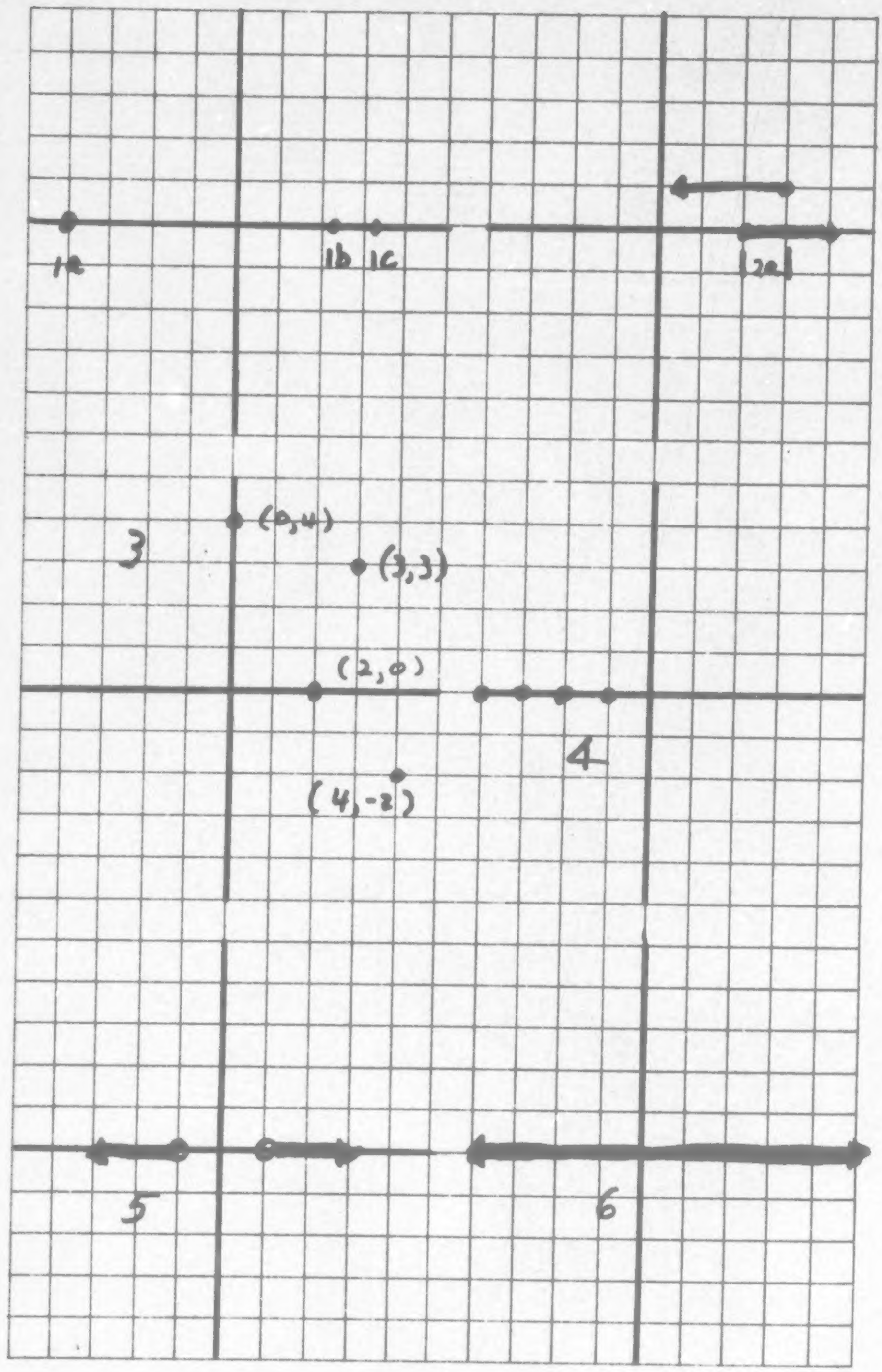
   $P = \{x \mid -4 \leq x < 6\}$

10. If your major is in a technical field, identify how the Cartesian Coordinate system has been employed or modified and applied to your particular field.

Performance Test Coordinates



Performance Test Answer Key


1a
1b
1c
2a
3
(0,4)
(3,3)
(2,0)
(4,-2)
4
5
6

Performance Test Answer Key (Continued)

7. (a) $x \leq -6$
   (b) $(-5, -2)$
   (c) $-3\frac{1}{2}$ or $(-3\frac{1}{2}, 0)$
   (d) $(2, -2)$
   (e) 2 or $(2, 0)$
   (f) $(2, 4)$
   (g) $x \geq 7$

8. ordered pair - $(3, -6)$
   abscissa - x-axis
   ordinate - y-axis
   origin - 0
   set - $p = \{x \mid x < 2\}$
   real number - π

9. P is the set of all x such that x is greater than or equal to -4 and x is less than 6.

10.

Technical Area	Application of Graphing
Automotive	Ignition Test Machinery
Nursing	Temperature charts
Machine Tool	Numerical Control
Electronics	Wave Oscilliscopes
Welding/Metallurgy	T-Curves, Stress-Strain
Drafting	Numerical Control

A-PAK #12
FACTORING

Rationale

Factoring algebraic polynomials provides one means of solving polynomial equations--linear, quadratic, and cubic. And while not all polynomial equations can be solved in this way (it may not be possible to factor the equation) factoring the equation for a solution is the initial technique since it is relatively quick.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Factor given polynomials containing the following factoring possibilities:
   (a) Common monomial factors.
   (b) The difference of two squares.
   (c) Trinomial squares.

2. Completely factor a given polynomial until the factors are prime.

3. Identify given polynomials as either:
   (a) Binomials.
   (b) Trinomials.

4. Develop a systemized method for treating a polynomial by factoring methods.

5. Check factoring accuracy by remultiplying factors in order to obtain the original expression.

Performance Activities

1. Attend the Factoring chalkboard session.

2. Read "Factoring" included in this A-PAK.

3. Study pages 14-17, 107-112 in Technical Mathematics.

4. Practice your factoring skills with the following problem sets: page 109, even problems 1-44; page 113, problems 1-3, 5, 15, 19.

5. Review A-PAK #6, SPECIAL PRODUCTS.

FACTORING

Common Monomial Factors

Numbers can often be expressed as the product of two or more factors. And when the factors can no longer be separated into smaller factors, they are termed prime factors.

Consider the number 12. It may be factored as:

$$12 = 2 \cdot 3 \cdot 2 \quad \text{or} \quad 2^2 \cdot 3$$

Note that neither the 2 or 3 may be further factored. Thus they are prime factors.

In algebra, it is also necessary to factor a polynomial. When factoring, the polynomial is subjected to a series of "possibilities" that may or may not prove feasible. The first possibility is recognized as the distributive property.

Factor: $6x + 12y = 6(x + 2y)$

Both 6 and $(x + 2y)$ are factors of $6x + 12y$. The 6 is called the monomial factor or common monomial factor of the polynomial since it is a factor of every term in the polynomial. Always search the polynomial first for any common monomial factors. If they are evident, they are factored out--removed. Examine the following and note how the common monomial factor is extracted:

$$3xy + 12x = 3x(y + 4)$$

$$5a^3 - 25a^2 + 10a = 5a(a^2 - 5a + 2)$$

$$x^2(a + b) + y(a + b) = (a + b)(x^2 + y)$$

$$ab + ac + ad = a(b + c + d)$$

$$3\sqrt{b} + 2\sqrt{b} = \sqrt{b}(3 + 2)$$

The Difference of Two Squares

The binomial $a^2 - b^2$ is factored $(a + b)(a - b)$. When the first term is a perfect square, the second term is a perfect square, and the two are separated by a negative sign, it is factorable. Examine the following and notice how the difference of two squares is factored:

$$a^2 - 9 = (a + 3)(a - 3)$$

$$a^2 - b^2c^2 = (a + bc)(a - bc)$$

$$3x^2 - 27 = 3(x^2 - 9) = 3(x + 3)(x - 3)$$

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$$

$$9(y - 1)^2 - 25 = [3(y - 1) + 5][3(y - 1) - 5)]$$

Trinomial Square

A trinomial square has the following pattern:

$a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$

The first term is a perfect square. The third term is a perfect square with a positive sign. The middle term, the product of the square root of the first and third terms, doubled.

In order to factor a trinomial square, use the sign of the middle term:

$$a^2 \underline{+ 2ab} + b^2 = (\quad + \quad)(\quad + \quad)$$

Find the square root of the first term:

$$\underline{a^2} + 2ab + b^2 = (a + \quad)(a + \quad)$$

Find the square root of the last term:

$$a^2 + 2ab + \underline{b^2} = (a + b)(a + b)$$

$$a^2 \underline{- 2ab} + b^2 = (\quad - \quad)(\quad - \quad)$$

$$\underline{a^2} - 2ab + b^2 = (a - \quad)(a - \quad)$$

$$a^2 - 2ab + \underline{b^2} = (a - b)(a - b)$$

Examine the following examples of factoring a trinomial square:

$$a^2 - 2a + 1 = (a - 1)(a - 1)$$

$$9x^2 + 6xy + y^2 = (3x + y)(3x + y)$$

$(25 + b - 10b$ = rearrange
(
$(25 - 10b + b^2 = (5 - b)(5 - b)$

Common monomial factor, trinomial square:

$$3a^2 + 6ab + 3b^2 = 3(a^2 + 2ab + b^2) = 3(a + b)(a + b)$$

Rearrange, factor trinomial square, factor difference of two squares:

$$x^2 + 2x + 1 - y^2 = \left[(x^2 + 2x + 1) - y^2\right] =$$

$$\left[(x + 1)(x + 1) - y^2\right] = \left[(x + 1)^2 - y^2\right] =$$

$$\left[(x + 1) + y\right]\left[(x + 1) - y\right] = (x + 1 + y)(x + 1 - y)$$

Performance Test

Factor the following expressions. If factorable, they must be factored prime. When the expression is unfactorable, write not factorable. Use your systemized approach to factoring and remultiply all factors of an expression in order to check your mathematical operations.

1. $-18a^4 - 6a^2$
2. $7a + 7b + 7c$
3. $6(a + b) + 12(a + b)^2$
4. $5ax + 10ay + 15a$
5. $ax + by + cz$
6. $4 - 4x + 4y$
7. $4x(2y + 5) + 7(2y + 5)$
8. $A^2 - B^2$
9. $64 - 25$
10. $a^2 + b^2$
11. $a^2 - 1$
12. $a^2 + 4a + 4$
13. $9a^2b^2 - 16$
14. $(a + b)^2 - 9$
15. $35 - x^2$
16. $a^2 - 2a + 4$
17. $x^4 - 16x^2 + 64$
18. $x^2 + 2xy + y^2$
19. $x + 2x + 4$
20. $-1 + x^2$

Identify the following by a B for binomial, T for trinomial, or M for monomial.

21. xyz^2
22. $a^2 + b$
23. $\frac{3a}{b}$
24. $a^2 + 2a + 4$
25. $xy - 2b$

Performance Test Answer Key

1. $-6a^2(3a^4 + 1)$
2. $7(a + b + c)$
3. $6(a + b)[1 + 2(a + b)]$
4. $5a(x + 2y + 3)$
5. Not factorable
6. $4(1 - x + y)$
7. $(2y + 5)(4x + 7)$
8. $(A + B)(A - B)$
9. $(8 + 5)(8 - 5)$
10. Not factorable
11. $(a + 1)(a - 1)$
12. $(a + 2)(a + 2)$
13. $(3ab + 4)(3ab - 4)$
14. $[(a + b) + 3][(a + b) - 3]$
15. Not factorable
16. $(a - 2)(a - 2)$
17. $(x^2 - 8)(x^2 - 8)$
18. $(x + y)(x + y)$
19. Not factorable
20. $(x + 1)(x - 1)$
21. M
22. B
23. M
24. T
25. B

A-PAK #13
MORE FACTORING

Rationale

Since you have completed A-PAK #12, you have at your disposal three different ways to factor a polynomial. There are also two additional types of factoring possibilities. A polynomial is subjected to a total of five treatments in a systematic fashion. Usually one of the five will apply, and the polynomial may then be solved.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Factor given polynomials containing the following factoring possibilities:
   (a) Binomial Cubes.
   (b) Quadratic trinomials.

2. Subject given polynomials to a systemized five-step treatment and factor them until they are prime.

3. Check factoring accuracy, remultiplying factors in order to obtain the original expression.

4. Write the classic five factoring possibilities.

5. Express a personal opinion about "insight" with respect to factoring.

Performance Activities

1. Attend the "Second Factoring" chalkboard session.

2. Read "Factoring Continuum" included in this A-PAK.

3. Study pages 109-113 in Technical Mathematics.

4. Practice your factoring skills with the following problem sets: page 109-110, odd problems 1-44; pages 113-114, problems 4, 6-10, 14.

5. Memorize the "Classic Five" included in this A-PAK.

FACTORING CONTINUUM

Sum or Difference of Binomial Cubes

When the expression is a binomial, it may be factored as (a) a common monomial factor or (b) the difference of two squares. The expression should first be checked for these possibilities. The third possibility (called binomial cubes) is:

$$a^3 + b^3 \quad \text{or } a^3 - b^3$$

The first term is a perfect cube, and the third term is a perfect cube. This type of expression is factored:

$$a^3 + b^3 = (\text{binomial})(\text{trinomial})$$

The signs are: $= (\quad + \quad)(\quad - \quad + \quad)$

The first factor is the cube root of each of the terms:

$$a^3 + b^3 = (a + b)(\quad - \quad + \quad)$$

What is left?

$$a^3 + b^3 = (a + b)(a^2 - \quad + b^2)$$

The center term of the trinomial is the product of the two terms in the binomial factor without considering the sign:

$$a^3 + b^3 = (a + b)(a^2 - \underset{\nearrow}{ab} + b^2)$$

$$a \cdot b = ab$$

Another method of determining the factors of a binomial cube is by assuming that in the expression

$$a^3 + b^3, \ (a + b) \text{ is a factor.}$$

If so: $\dfrac{a^3 + b^3}{(a + b)}$ should yield an answer without a remainder when it is divided.

Note: $\dfrac{a^3 + b^3}{(a + b)} = (a^2 - ab + b^2)$

Examine the following binomial cubes and note how they are factored:

$$a^3 - 1 = (a - 1)(a^2 + a + 1)$$

$$8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$(a + b)^3 - 27 = \left[(a + b) - 3\right]\left[(a + b)^2 + 3(a + b) + 9\right]$$

Quadratic Trinomials

When the expression is a trinomial and not a trinomial square or does not contain a common monomial factor, it is assumed that the expression will factor. It must be subjected to a trial and error treatment.

If the sign of the third term is negative, then:

$$x^2 + x \underline{- 6} = (\ +\ )(\ -\ )$$

If the sign of the third term is positive, then examine the sign of the middle term; it determines what both signs will be:

$$x^2 \underline{+ 5x} + 6 = (\ +\ )(\ +\ )$$

$$x^2 \underline{- 5x} + 6 = (\ -\ )(\ -\ )$$

Note:

$$x^2 + x - 6 = (\ +\ )(\ -\ )$$

The factors of the first term are (1x)(1x). Thus:

$$x^2 + x - 6 = (x +\ )(x -\ )$$

The factors of the last term are (6)(1), (1)(6), (2)(3), (3)(2). What combination works?

$$x^2 + x - 6 = (x + 3)(x - 2)$$

Factoring:

$$x^2 + 5x + 6 = (\ +\ )(\ +\ )$$

$$(x +\ )(x +\ )$$

$$(x + 2)(x + 3)$$

Factoring:

$$x^2 - 5x + 6 = (\quad - \quad)(\quad - \quad)$$

$$(x - \quad)(x - \quad)$$

$$(x - 3)(x - 2)$$

Peruse the following examples of factoring quadratic trinomials:

$$a^2 + 3ab - 10b^2 = (a + 5)(a - 2)$$

$$5a^2 - 7a + 2 = (5a - 2)(a - 1)$$

$$8a^2 - 27a - 20 = (a - 4)(8a + 5)$$

$$3x^2 - 17ab - 6b^2 = (3a + 1b)(a - 6b)$$

THE CLASSIC FIVE

I. Common monomial factor.

$$ax \pm ay \pm az = a(x \pm y \pm Z)$$

II. Difference of two squares.

$$a^2 - b^2 = (a + b)(a - b)$$

III. Trinomial square.

$$a^2 \mp 2ab + b^2 = (a \pm b)(a \pm b)$$

IV. Sum or difference of binomial cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

V. Quadratic trinomial.

$$ABx^2 \pm (Ab + aB)x \pm ab = (Ax \pm a)(Bx \pm b)$$

Performance Test

Factor the following expressions. If factorable, they must be factored prime. When the expression is unfactorable, write not factorable. Also subject each problem to "The Classic Five" in a systemized fashion.

1. Identify each of the following:

   (a) $ax + ay + az = a(x + y + z)$

   (b) $a^2 - b^2 = (a + b)(a - b)$

   (c) $a^2 - 2ab + b^2 = (a - b)(a - b)$

   (d) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

   (e) $2x^2 + x - 6 = (2x - 3)(x + 2)$

2. $a^3 + b^3 =$

3. $x^3y^3 - 27 =$

4. $125x^6 + 8y^3c^9 =$

5. $7x^2 - 7 =$

6. $3x^2 + x + 1 =$

7. $24x^2 - 13y - 2 =$

8. $8a^2 - 34a + 33 =$

9. $a^2 + c^2 =$

10. $3y^2 - 14y - 5 =$

Re-multiply the factors in order to check your factoring accuracy.

11. If you have experienced a factoring "insight," express your opinion about the experience.

Performance Test Answer Key

1. (a) Common monomial factor.
   (b) Difference of two squares.
   (c) Trinomial square.
   (d) Sum or difference of cubes.
   (e) Quadratic trinomial.

2. $(a + b)(a^2 - ab + b^2)$

3. $(xy - 3)(x^2y^2 + 3xy + 9)$

4. $(5x^2 + 2yc^3)(25x^4 - 10x^2yc^3 + 4y^2c^6)$

5. $7(x^2 - 1) = 7(x + 1)(x - 1)$

6. Not factorable.

7. Not factorable.

8. $(4a - 11)(2a - 3)$

9. Not factorable.

10. $(3y + 1)(y - 5)$

11. Insight can be described as a sudden realization of the solution to a factoring puzzle. It is generally experienced by the problem solver when he first views the problem. It also is described as an exhilarating experience.

    Insight is not completely understood by educators or psychologists.

A-PAK #14
QUADRATIC EQUATIONS

Rationale

To date, our capability of solving equations has been restricted to equations of the first degree where the variable x is only to power 1. Factoring provides a method for solving quadratic equations where the variable (x^2) is of the second degree. Unfortunately, not all quadratic polynomials will factor, albeit they are solvable by other methods, one being a method called "completing the square."

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Apply factoring methods to given quadratic equations and solve them.

2. Specify the real number property applied to a factored quadratic equation that permits its solutions to be determined.

3. Identify the different kinds of quadratic equations and write their names.

4. Solve quadratic equations by the method called "completing the square."

Performance Activities

1. Attend the lecture on Quadratic Equations.

2. View "Quadratic Equations," Eye-Gate 5-6E, at the Center for Independent Study.

3. Read "Solving Quadratic Equations" included in this A-PAK.

4. Study pages 240-247 in Technical Mathematics.

5. Practice your skills with the following problem sets: page 243, problems 1-24; page 245, problems 1-25.

SOLVING QUADRATIC EQUATIONS

Factoring

Quadratic equations, such as $a^2 + 2a + 1 = 0$ or $a^2 = 8a - 15$ are second degree equations that can often be solved by factoring.

Example 1: $a^2 = 8a - 15$

$$a^2 - 8a + 15 = 0 \quad \text{(rearranged)}$$

$$(a - 5)(a - 3) = 0 \quad \text{(factored)}$$

Consider for a moment the following real number property:

$$a \cdot b = 0$$

There are three possibilities for a and b:

1. $a = 0$, $b =$ a number
2. $a =$ a number, $b = 0$
3. Both $a = 0$ and $b = 0$

We disregard the third possibility since the equation would be canceled. Then with the equation:

$$(a - 5)(a - 3) = 0$$

We assume first that $a - 5 = 0$

Thus: $a = 5$

We then assume $a - 3 = 0$

Thus: $a = 3$

The solution is (5, 3).

Note that both values will mathematically satisfy the equation:

$$a^2 - 8a + 15 = 0$$

$$5^2 - 8(5) + 15 = 0 \quad \text{(substituting 5)}$$

$$25 - 40 + 15 = 0$$

$$0 = 0$$

$a^2 - 8a + 15 = 0$

$3^2 - 8(3) + 15 = 0$ (substituting 3)

$9 - 27 + 15 = 0$

$0 = 0$

Example 2:

$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$ (factor)

$x - 2 = 0$ (assume)

$x = 2$ (solution)

Some quadratic equations are incomplete.

Example 3: $x^2 = 81$

$x^2 - 81 = 0$

$(x + 9)(x - 9) = 0$

$x + 9 = 0$ and $x - 9 = 0$

$x = -9$ $x = 9$

Thus the solution is $(-9, 9)$.

Example 4:

$4x^2 - 3x = 0$ (common monomial factor)

$x(4x - 3) = 0$

$x = 0$ and $4x - 3 = 0$

$4x = 3$

$x = \frac{3}{4}$

Thus the solution is $(0, \frac{3}{4})$.

Example 5:
$$(3x + 1)^2 = 0$$
$$(3x + 1)(3x + 1) = 0$$
$$3x + 1 = 0$$
$$3x = -1$$
$$x = -\frac{1}{3}$$

A quadratic equation will have two possible answers that:

(a) may be identical (same answer).
(b) may be + or -.
(c) one will be a number; the other a zero.

Completing the Square

When the quadratic equation is not complete, it may be solved by a method known as "completing the square."

Consider: $2x^2 - 6x = 1$

$$\frac{2x^2}{2} - \frac{6x}{2} = \frac{1}{2}$$

(reduce the numerical coefficient
(of the x^2 term to 1 by dividing by 2.

$$x^2 - 3x = \frac{1}{2}$$

$$x^2 - 3x = \frac{1}{2}$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = \frac{1}{2} + \left(\frac{3}{2}\right)^2$$

(Divide the numerical co-
(efficient of the second
(term by 2 and add its
(square to both members
(of the equation.

We have created a trinomial square and thus it must factor:

$$\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = \frac{1}{2} + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = \frac{11}{4}$$

$$\sqrt{\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right)} = \sqrt{\frac{11}{4}}$$

Extract the square root of each member of the equation.

$$x - \frac{3}{2} = \pm\sqrt{\frac{11}{4}}$$

The root of $\sqrt{\frac{11}{4}}$ may be either positive or negative.

$$x - \frac{3}{2} = + \sqrt{\frac{11}{4}}$$ Solve for the positive.

$$x = \frac{3}{2} + \sqrt{\frac{11}{4}} = \frac{3}{2} + 1.6 = 3.1$$

$$x = \frac{3}{2} = - \sqrt{\frac{11}{4}}$$ Solve for the negative.

$$x = \frac{3}{2} - \sqrt{\frac{11}{4}} = \frac{3}{2} - 1.6 = -.1$$

Thus the solution is (3.1, -.1).

The answer may be presented:

$$x = \frac{3}{2} + \sqrt{\frac{11}{4}} = \frac{3}{2} + \frac{\sqrt{11}}{2} = \frac{3 + \sqrt{11}}{2}$$

$$x = \frac{3}{2} - \sqrt{\frac{11}{4}} = \frac{3}{2} - \frac{\sqrt{11}}{2} = \frac{3 - \sqrt{11}}{2}$$

The solution is $\left(\frac{3 + \sqrt{11}}{2}, \frac{3 - \sqrt{11}}{2} \right)$

Example 2:

$$2x^2 - 12x - 36 = 0$$

$$\frac{2x^2}{2} - \frac{12x}{2} = \frac{36}{2}$$ Rearrange.

$$x^2 - 6x = 18$$ Reduce numerical coefficient of the first term to 1 by dividing by 2.

$$x^2 - 6x + (3)^2 = 18 + (3)^2$$ Divide the numerical coefficient of the second term by 2 and add its square to both members of the equation.

$$(x - 3)(x - 3) = 27$$ Factor.

$$\sqrt{(x - 3)(x - 3)} = \sqrt{27}$$ Find roots.

$$x - 3 = \pm \sqrt{27}$$

$$x = 3 \pm \sqrt{27}$$

The solution is $(3 + \sqrt{27}, 3 - \sqrt{27})$.

Performance Test

1. Write mathematically the real number property that permits the solving of a quadratic equation by factoring and list the possibilities under the property.

2. Identify each of the following types of quadratic equations by name:

   (a) $2x^2 + 9x = 0$

   (b) $x^2 - 36 = 0$

   (c) $x^2 + 2x + 1 = 0$

Factor the following quadratic equations by first factoring them and solving for the roots:

3. $x^2 - 49 = 0$
4. $a^2 - 4a = 21$
5. $x^2 + 1 = 2x$
6. $x^2 = -2x$
7. $y^2 + y - 6 = 0$
8. $(x - 2)(x + 1) = x(2 - x)$
   (multiply, collect, factor, solve).

Solve the following by completing the square:

9. $x^2 - 8x \quad = 0$
10. $x^2 - x - 3 = 0$

Performance Test Answer Key

1. $a \cdot b = 0$; then either $a = 0$, $b = 0$, or both a and $b = 0$.

2. (a) *Incomplete quadratic equation* since the third term is missing.
   (b) *Incomplete quadratic equation* since the second term is missing.
   (c) *Complete quadratic equation* since all terms are present.

3. $(7, -7)$

4. $(7, -3)$

5. $(1, 1)$

6. $(0, -2)$

7. $(-3, 2)$

8. (a) $x^2 - x - 2 = 2x - x^2$

   (b) $2x^2 - 3x - 2 = 0$

   (c) $(2x + 1)(x - 2) = 0$

   $2x + 1 = 0 \qquad x - 2 = 0$

   $x = -\frac{1}{2} \qquad x = 2$

   $(-\frac{1}{2}, 2)$

9.
$$x^2 - 8x = 0$$
$$x^2 - 8x + (4)^2 = (4)^2$$
$$\sqrt{(x - 4)(x - 4)} = \sqrt{16}$$
$$x - 4 = \pm 4$$
$$x = 8$$
$$x = 0$$
$$(8, 0)$$

10. $x^3 - x = 3$
$$x^2 - x + \left(\frac{1}{2}\right)^2 = 3 + \left(\frac{1}{2}\right)^2$$
$$\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{13}{4}$$
$$x - \frac{1}{2} = \frac{\pm\sqrt{13}}{2}$$
$$x = \frac{1 + \sqrt{13}}{2}$$
$$x = \frac{1 - \sqrt{13}}{2}$$

A-PAK #14A (OPTIONAL)
APPLYING THE QUADRATIC FORMULA

Rationale

Some quadratic equations do not factor, and applying the method of completing the square is cumbersome. Completing the square, however, can be utilized to derive a quadratic formula that in turn will permit any quadratic equation (providing the discriminant is not negative) to be solved. Deriving the quadratic formula is also in itself an exercise that will stimulate the mathematical palate.

Instructional Objectives*

After completing this A-PAK, the learner will be able to:

1. Understand how the quadratic formula is derived.

2. Apply the quadratic formula for solving quadratic equations.

Performance Activities

1. Read "The Quadratic Formula" included in this A-PAK.

2. Study pages 247-252 in Technical Mathematics.

3. Practice solving quadratic equations with the quadratic formula, page 250, problems 5-27, odd numbers only.

Performance Test

There will not be any Performance Test for this A-PAK. It is hoped that the exercise has increased your knowledge and expanded your horizons.

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*This is an optional A-PAK; therefore, the objectives are not specified as measurable.

THE QUADRATIC FORMULA

Completing the square for the classic quadratic equation, we derive:

$$ax^2 + bx + c = 0 \qquad \text{Classic form.}$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Dividing by a.}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Completing square.}$$

$$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{Factoring.}$$

$$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = \frac{b^2 - 4ac}{4a^2} \qquad \text{Adding fractions.}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Extracting roots.}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Solving for x.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Adding fractions.}$$

The $b^2 - 4ac$ is called the discriminant of the quadratic equation and must be non-negative.

We can solve any quadratic equation by means of the quadratic formula providing the discriminant is greater than zero or equal to zero by means of the following method:

Problem 1:

$$2x^2 + 3x + 1 = 0$$

Let:

$$\underset{a}{2x^2} + \underset{b}{3x} + \underset{c}{1} = 0$$

And:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting:

$$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(1)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{1}}{4} = \frac{-3 \pm 1}{4}$$

Solving for +:

$$x = -\frac{2}{4} = -\frac{1}{2}$$

Solving for -:

$$x = \frac{-4}{4} = -1$$

$$\left(-\frac{1}{2}, -1\right)$$

Problem 2:

$$\underset{a}{3x^2} + \underset{b}{x} - \underset{c}{1} = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(1) \pm \sqrt{1^2 - 4(3)(-1)}}{2(3)}$$

$$x = \frac{-1 \pm \sqrt{13}}{6}$$

$$x = \frac{-1 + \sqrt{13}}{6} \qquad x = \frac{-1 - \sqrt{13}}{6}$$

$$\left(\frac{-1 + \sqrt{13}}{6}, \frac{-1 - \sqrt{13}}{6}\right)$$

A-PAK #15
SYSTEMS OF EQUATIONS

Rationale

Some linear equations contain two variables. The solution set can be determined providing a system is evident. A system is composed of two equations if there are two variables or three equations if there are three variables. Also, the solution (an ordered pair) will satisfy both equations.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Solve given equation systems by means of addition.

2. Solve given equation systems by means of subtraction.

3. Solve given equation systems by means of substitution.

4. Write the solution set for equation systems as ordered pairs, i.e., (x, y).

5. Check the solution of solved systems by the substitution of solutions into the original equations.

6. Demonstrate orderly methodology when solving systems by labeling methods and logical problem solving.

Performance Activities

1. Attend the chalkboard lecture "Simultaneous Equations."

2. Read "Linear Equation Systems" included in this A-PAK.

3. Study pages 208-213 in Technical Mathematics.

4. Practice solving systems with the following problem sets: page 213, problems 1-24. Solve first by the addition/subtraction method and check your solutions by the substitution method of solving.

LINEAR EQUATION SYSTEMS

A system of equations contains two equations as well as two variables. The solution to the system (solution set) is an ordered pair that is the solution for each equation. (It must satisfy both equations.) The solution set of the system is also the intersection of the system. Systems are sometimes called simultaneous equations, and the system may be solved several ways.

Addition or Subtraction Method

Example 1: $3x + 2y = 4$ Equation 1

$\underline{2x + 5y = 10}$ Equation 2

Multiply the first equation by 2 and the second equation by 3.

$$\begin{array}{rr} & 6x + 4y = 8 \\ \text{Subtract:} & \underline{(-)\ 6x + 15y = 30} \\ & -11y = 22 \\ & y = 2 \end{array}$$

Substitute the value for y into either of the equations in order to solve for x.

$6x + 4y = 8$ Equation 1

$$6x + 4(2) = 8$$

$$6x + 8 = 8$$

$$6x = 8 - 8$$

$$6x = 0$$

Note that $x = 0$ and $y = 2$ satisfies both original equations of the system. The solution set is specified as an ordered pair (x, y) or $(0, 2)$.

Example 2: $y = 2x + 7$

$$x + y = 10$$

Rearrange: $-2x + y = 7$ Equation 1

$\underline{x + y = 10}$ Equation 2

Multiply Equation 2 by 2:

Add:

$$\begin{array}{r} -2x + y = 7 \\ \underline{2x + 2y = 20} \\ 3y = 27 \\ \\ y = 9 \end{array}$$

Substitute the value for y into either equation in order to solve for x.

$$\begin{array}{rl} -2x + y = 7 & \text{Equation 1} \\ -2x + 9 = 7 & \\ -2x = 7 - 9 & \\ -2x = -2 & \\ x = 1 & \\ (1, 9) & \end{array}$$

Example 3:

$$\begin{array}{rl} 2x - y = 6 & \text{Equation 1} \\ \underline{-3x + 2y = -8} & \text{Equation 2} \end{array}$$

Multiply equation 1 by 2:

Add:

$$\begin{array}{r} 4x - 2y = 12 \\ \underline{-3x + 2y = -8} \\ x = 4 \end{array}$$

Substitute the value for x into either of the equations in order to solve for y.

$$\begin{array}{rl} -3x + 2y = -8 & \text{Equation 2} \\ -3(4) + 2y = -8 & \\ -12 + 2y = -8 & \\ 2y = 4 & \\ y = 2 & \\ (4, 2) & \end{array}$$

Substitution Method

The substitution method for solving systems involves solving for one variable in terms of the other variable (x in terms of y or y in terms of x); substituting the expression in the other equation; and then solving either equation for the second variable.

Example 1: $3x + 2y = 4$ Equation 1
$2x + 5y = 10$ Equation 2

Solving for x in terms of y: $3x + 2y = 4$ Equation 1

$$3x = 4 - 2y$$

$$x = \frac{4}{3} - \frac{2y}{3}$$

Substituting: $2x + 5y = 10$ Equation 2

$$2\left(\frac{4}{3} - \frac{2y}{3}\right) + 5y = 10$$

$$\frac{8}{3} - \frac{4y}{3} + 5y = 10$$

$$8 - 4y + 15y = 30$$

$$11y = 30 - 8$$

$$11y = 22$$

$$y = 2$$

Substituting y = 2 in either equation: $3x + 2y = 4$ Equation 1

$$3x + 2(2) = 4$$

$$3x + 4 = 4$$

$$3x = 0$$

$$x = 0$$

$$(0, 2)$$

Example 2: $y = 2x + 7$ Equation 1

$x + y = 10$ Equation 2

Equation 1 has a value for y in terms of x.

Substituting: $x + y = 10$ Equation 2

$x + (2x + 7) = 10$

$3x + 7 = 10$

$3x = 3$

$x = 1$

Substituting
x = 1 in
either
equation: $x + y = 10$ Equation 2

$1 + y = 10$

$y = 9$

$(1, 9)$

Example 3: $2x - y = 6$ Equation 1

$-3x + 2y = -8$ Equation 2

Solving for x
in terms of y: $2x - y = 6$ Equation 1

$2x = 6 + y$

$x = 3 + \frac{y}{2}$

Substituting: $-3x + 2y = -8$ Equation 2

$-3(3 + \frac{y}{2}) + 2y = -8$

$-9 - \frac{3y}{2} + 2y = -8$

$-18 - 3y + 4y = -16$

$y = -16 + 18$

$y = 2$

Substituting
y = 2 in
either
equation: $2x - y = 6$ Equation 1

$2x - 2 = 6$

$2x = 8$

$x = 4$

$(4, 2)$

Performance Test

Solve the following systems by means of the addition method. Write the solution as an ordered pair in x and y. Check the solution by substitution with Equation 1.

1. $3x - y = 9$ Equation 1
   $\underline{x + 2y = 10}$ Equation 2

2. $3x + 2y = 11$ Equation 1
   $\underline{5x - 3y = 31}$ Equation 2

Solve the following equations by means of the subtraction method. Write the solution as an ordered pair in a and b. Check the solution by substitution with Equation 2.

3. $2a + 3b = 4$ Equation 1
   $\underline{2a + 6b = 10}$ Equation 2

4. $7a - 2b = 8$ Equation 1
   $\underline{7a + 3b = -12}$ Equation 2

Solve problems 5 and 6 by the substitution method; label each equation.

5. $3x - y = 9$ Equation 1
   $x + 2y = 10$ Equation 2

6. $3x + 2y = 11$ Equation 1
   $5x - 3y = 31$ Equation 2

7. Do your answers for problems 5 and 6 check out with problems 1 and 3 of this test?

Performance Test Answer Key

1.

$$\begin{array}{rcr} (3x - y = 9)2 & = & 6x - 2y = 18 \\ \underline{x + 2y = 10} & & \underline{x + 2y = 10} \\ & & 7x = 28 \\ & & x = 4 \end{array}$$

$$\begin{array}{rl} 6x - 2y = 18 & \\ 6(4) - 2y = 18 & \\ 24 - 2y = 18 & \qquad (4, 3) \\ -2y = -6 & \\ y = 3 & \end{array}$$

Check:

$$\begin{array}{r} 3x - y = 9 \\ 3(4) - 3 = 9 \\ 12 - 3 = 9 \\ 9 = 9 \end{array}$$

2.

$$\begin{array}{rcr} (3x + 2y = 11)3 & = & 9x + 6y = 33 \\ \underline{(5x - 3y = 31)2} & = & \underline{10x - 6y = 62} \\ & & 19x = 95 \\ & & x = 5 \end{array}$$

$$\begin{array}{rl} 5x - 3y = 31 & \\ 5(5) - 3y = 31 & \\ -3y = 6 & \qquad (5, -2) \\ y = -2 & \end{array}$$

Check:

$$\begin{array}{r} 3x + 2y \quad = 11 \\ 3(5) + 2(-2) = 11 \\ 15 - 4 = 11 \\ 11 = 11 \end{array}$$

3.

$$\begin{array}{rcr} 2a + 3b = 4 & = & 2a + 3b = 4 \\ \underline{2a + 6b = 10} & = & \underline{-2a - 6b = -10} \\ & & -3b = -6 \\ & & b = 2 \end{array}$$

$$\begin{array}{rl} 2a + 3b = 4 & \\ 2a + 3(2) = 4 & \\ 2a = -2 & \qquad (-1, 2) \\ a = -1 & \end{array}$$

Check:

$$\begin{array}{r} 2a + 6b \quad = 10 \\ 2(-1) + 6(2) = 10 \\ -2 + 12 = 10 \\ 10 = 10 \end{array}$$

4.

$$\begin{array}{rcr} 7a - 2b = 8 & = & 7a - 2b = 8 \\ \underline{7a + 3b = -12} & = & \underline{-7a - 3b = 12} \\ & & -5b = 20 \\ & & b = -4 \end{array}$$

$$\begin{aligned} 7a - 2b &= 8 \\ 7a - 2(-4) &= 8 \\ 7a &= 0 \qquad (0, -4) \\ a &= 0 \end{aligned}$$

Check: $$\begin{aligned} 7a + 3b &= -12 \\ 7(0) + 3(-4) &= -12 \\ -12 &= -12 \end{aligned}$$

5.

$$\begin{aligned} 3x - y &= 9 \qquad \text{Equation 1} \\ x + 2y &= 10 \qquad \text{Equation 2} \end{aligned}$$

$$\begin{aligned} 3x - y &= 9 \qquad \text{Equation 1} \\ 3x &= y + 9 \\ x &= \frac{y}{3} + 3 \end{aligned}$$

$$\begin{aligned} x + 2y &= 10 \qquad \text{Equation 2} \\ \frac{y}{3} + 3 + 2y &= 10 \\ y + 9 + 6y &= 30 \\ 7y &= 21 \\ y &= 3 \qquad (4, 3) \end{aligned}$$

$$\begin{aligned} x + 2y &= 10 \\ x + 2(3) &= 10 \\ x + 6 &= 10 \\ x &= 4 \end{aligned}$$

6.

$$\begin{aligned} 3x + 2y &= 11 \qquad \text{Equation 1} \\ 5x - 3y &= 31 \qquad \text{Equation 2} \end{aligned}$$

$$\begin{aligned} 3x + 2y &= 11 \qquad \text{Equation 1} \\ 2y &= 11 - 3x \\ y &= \frac{11}{2} - \frac{3}{2}x \end{aligned}$$

$$\begin{aligned} 5x - 3\left(\frac{11}{2} - \frac{3}{2}x\right) &= 31 \qquad \text{Equation 2} \\ 5x - \frac{33}{2} + \frac{9}{2}x &= 31 \\ 10x - 33 + 9x &= 62 \\ 19x &= 95 \\ x &= 5 \qquad (5, -2) \end{aligned}$$

$$\begin{aligned} 5x - 3y &= 31 \\ 5(5) - 3y &= 31 \\ 25 - 3y &= 31 \\ -36 &= 6 \\ y &= -2 \end{aligned}$$

7. Yes, problem 5 is problem 1 and problem 6 is problem 3.

A-PAK #16
GRAPHING EQUATIONS

Rationale

Graphing equations is an integral part of contemporary algebra and higher mathematics, for not only is a visual picture of the equation then evident but graphing a system is also a method of solving systems. This A-PAK builds upon the technique developed in A-PAK #11, "Graphing Numbers, Sets, and Ordered Pairs."

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Chart and graph given functions.
2. Chart and graph given quadratic equations.
3. Chart and graph systems of given equations.
4. Determine from specified equations:
   (a) The y-intercept.
   (b) The slope--positive or negative.
5. Determine from a graph if a system is:
   (a) Independent (and show solution).
   (b) Inconsistent.
   (c) Dependent.

Performance Activities

1. Attend the regular class time session on Overhead Projector Graphing.

2. Read "Graphing Quadratic Equations" included in this A-PAK.

3. Read "Graphing Systems" included in this A-PAK.

4. View "Graphic Solution of Linear Equations and Linear Inequalities," Eye-Gate 5-6D; "Graphs of Quadratic Functions," Eye-Gate 5-6G; and "Graphic Solution of Inequalities and Equalities," Eye-Gate 5-6H.

5. Study pages 254-255, 267-269, 205-208 in Technical Mathematics.

6. Practice your skills with the following problem sets: page 269, problems 1, 5, 7, 8, 11; page 213, problems 1-6.

7. Read pages 36-48, "Cartesian Plane," in Concepts of Sets.

GRAPHING QUADRATIC EQUATIONS

Functions

The expression

$$f : x \text{ --- } 3x + 1$$

is read: the function f that pairs x with $3x + 1$. A function is two sets tied together with a rule that links one element of a set together with an element of another set.

If:	x	is	$3x + 1$
and x =	1		4
and x =	2		7
and x =	-2		5

Example 1:

When charting a quadratic equation like $y = -3x^2$, we establish values for x and determine $-3x^2$, which is equal to y.

x	$-3x^2$	$= y$	(x, y)
0	$-3(0)^2$	0	$(0, 0)$
1	$-3(1)^2$	-3	$(1, -3)$
$\frac{1}{2}$	$-3\left(\frac{1}{2}\right)^2$	$-\frac{3}{4}$	$\left(\frac{1}{2}, -\frac{3}{4}\right)$
-1	$-3(-1)^2$	-3	$(-1, -3)$
$-\frac{1}{2}$	$-3\left(-\frac{1}{2}\right)^2$	$-\frac{3}{4}$	$\left(-\frac{1}{2}, -\frac{3}{4}\right)$

Note that any values for x may be substituted as elements. The values for y are tied by the rule or equation $y = -3x^2$. Thus an ordered pair results. The ordered pairs can be graphed. The points form a curve known as a parabola (see graph $y = 3x^2$).

Example 2: Chart and graph $y = x^2 - 2x - 3$

x	$x^2 - 2x - 3$	$= y$	(x, y)
0	$0^2 - 2(0) - 3$	-3	(0, -3)
1	$1^2 - 2(1) - 3$	-4	(1, -4)
2	$2^2 - 2(2) - 3$	-3	(2, -3)
-3	$3^2 - 2(3) - 3$	0	(3, 0)
-4	$-4^2 - 2(4) - 3$	5	(-4, 5)
-2	$(-2)^2 - 2(-2) - 3$	5	(-2, 5)
-1	$(-1)^2 - 2(-1) - 3$	0	(-1, 0)

(See graph $x^2 - 2x - 3$)

Example 3: Chart and graph $x^2 - 4x + 4$

x	$x^2 - 4x + 4$	$= y$	(x, y)
0	$(0)^2 - 4(0) + 4$	4	(0, 4)
1	$(1)^2 - 4(1) + 4$	1	(1, 1)
2	$(2)^2 - 4(2) + 4$	0	(2, 0)
3	$(3)^2 - 4(3) + 4$	1	(3, 1)
4	$(4)^2 - 4(4) + 4$	4	(4, 4)

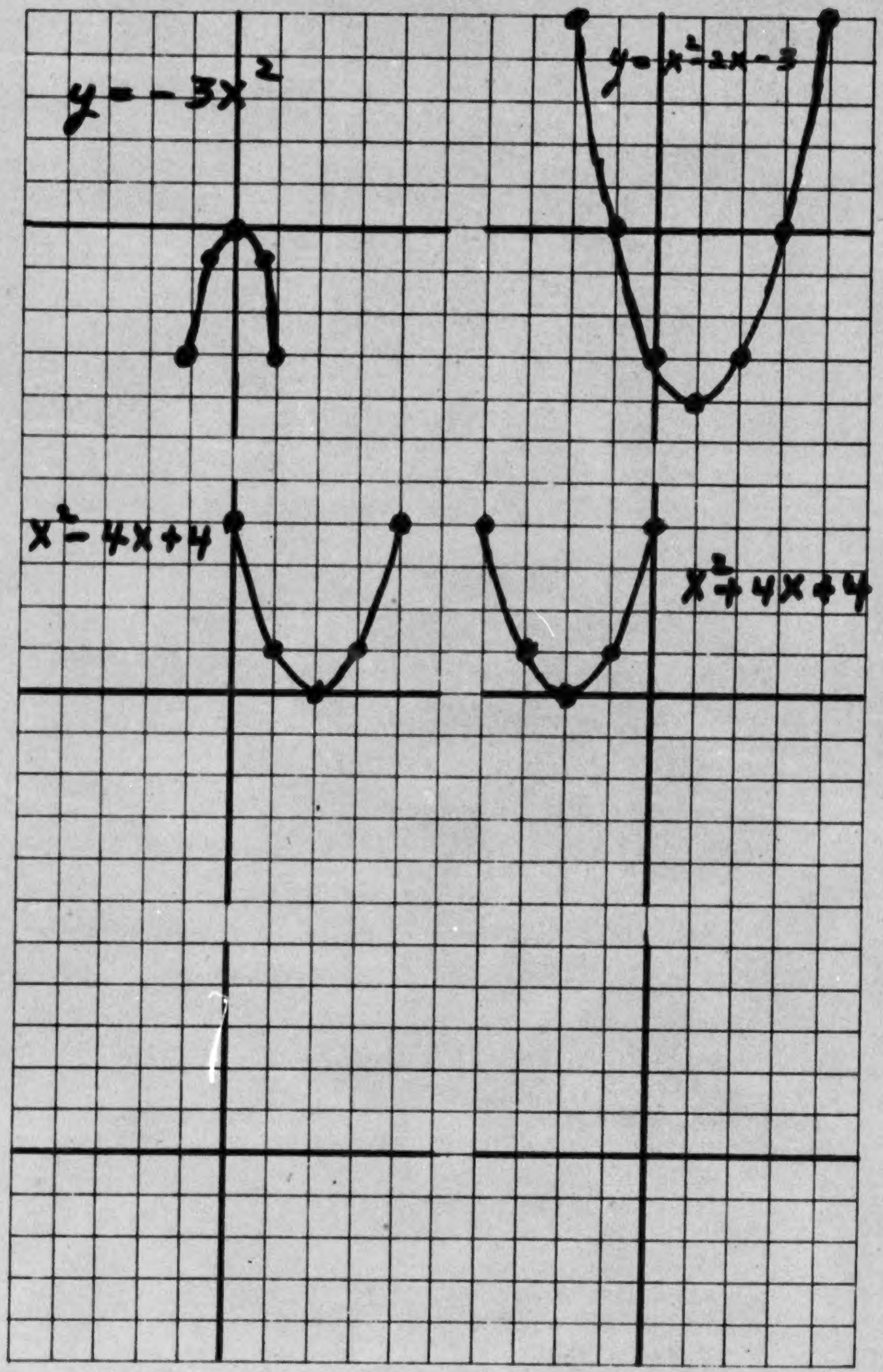
(See graph $x^2 - 4x + 4$)

Example 4: Chart and graph $x^2 + 4x + 4$

x	$x^2 + 4x + 4$	$= y$	(x, y)
0	$(0)^2 + 4(0) + 4$	4	(0, 4)
-1	$(-1)^2 + 4(-1) + 4$	1	(-1, 1)
-2	$(-2)^2 + 4(-2) + 4$	0	(-2, 0)
-3	$(-3)^2 + 4(-3) + 4$	1	(-3, 1)
-4	$(-4)^2 + 4(-4) + 4$	4	(-4, 4)

(See graph $x^2 + 4x + 4$)

GRAPHS



GRAPHING SYSTEMS

Linear Equations

A linear equation in two variables, such as $2x + 4y = 8$ will graph as a line. The equation is of the form $y = mx + b$ where m = slope and b = the y intercept.

Example 1: Graph $2x + 4y = 8$

Step 1: Solve for y in terms of x.

$$4y = -2x + 8$$

$$y = -\frac{1}{2}x + 2$$

$$y = mx + b \quad \text{(note the form)}$$

Step 2: Substitute values for x. Always determine three points.

x	$-\frac{1}{2}x + 2$	$= y$	(x, y)
0	$-\frac{1}{2}(0) + 2$	2	(0, 2)
2	$-\frac{1}{2}(2) + 2$	1	(2, 1)
-2	$-\frac{1}{2}(-2) + 2$	3	(-2, 3)

Step 3: Graph the ordered pairs (see graph $y = -\frac{1}{2}x + 2$).

In $y = mx + b$, b is the y intercept. Note that when $x = 0$, $b = 2$ is easily identified when the equation $2x + 4y = 8$ is transformed to $y = -\frac{1}{2}x + 2$. The line crosses the y axis at (0, 2).

Note in the equation $y = -\frac{1}{2}x + 2$:

$-\frac{1}{2}$ corresponds to m in $y = mx + b$ which is the slope of the line. If negative, the line will slope \ but if positive, the line will slope/.

Slope is further defined as:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{y}{x}$$

In our equation $m = -\frac{1}{2}$, assume then $\frac{-1}{2} = \frac{\text{rise}}{\text{run}}$

This means that from an established point on the line for every -1 on the y axis there will be a 2 on the x axis. Note the slope increments on graph $-\frac{1}{2}x$

If we assume $\frac{1}{-2} = \frac{\text{rise}}{\text{run}}$, for every +1 on the y axis there will be a - 2 on the x axis.

Example 2: Graph $2x - y = 7$

Step 1: Solve for y in terms of x.

$$-y = -2x + 7$$

$$y = 2x - 7$$

Step 2: Substitute values for x.

x	2x - 7	=	y	(x, y)
0	2(0) - 7		-7	(0, -7)
2	2(2) - 7		-3	(2, -3)
3	2(3) - 7		-1	(3, -1)

Step 3: Graph the ordered pairs (see graph $y = 2x - 7$).

The y intercept is (0, -7); the slope is $m = \frac{2}{1}x$. Therefore the slope is / or positive. For every rise of 2, there is a run of 1.

A line may similarly be graphed for the x intercept.

$$2x - y = 7$$

$$2x = y + 7$$

$$x = \frac{1}{2}y + \frac{7}{2}$$

If $y = 0$, then $x = \frac{7}{2}$ or $(\frac{7}{2}, 0)$.

Example 3: Graph y = 4

Step 1: Solve for y in terms of x.

$$y = 0x + 4$$

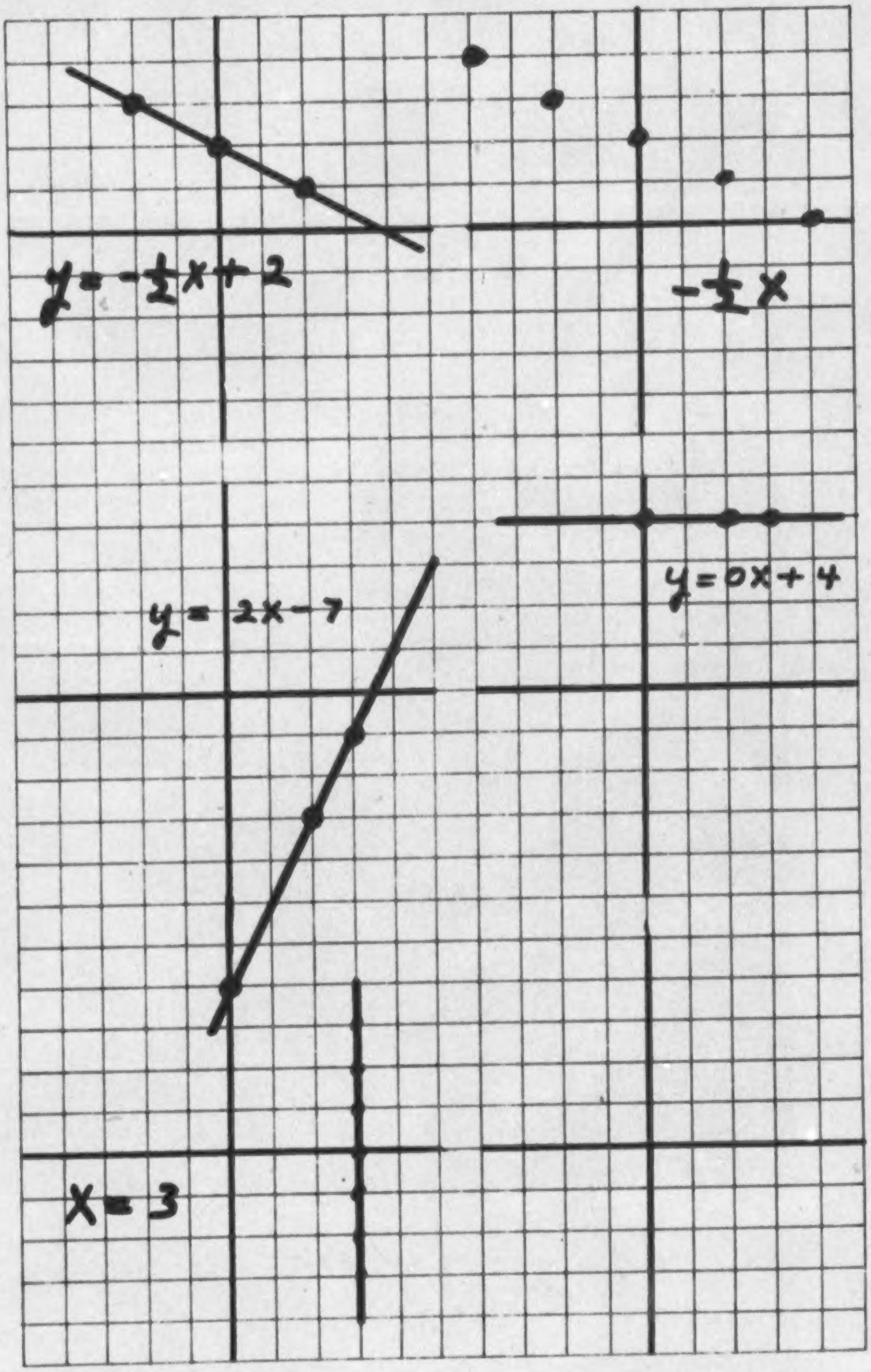
Step 2: Substitute values for x.

x	0x + 4 =	y	(x, y)
1	0(1) + 4	4	(1, 4)
2	0(2) + 4	4	(2, 4)
3	0(3) + 4	4	(3, 4)

From the equation y = 0x + 4, we determine the y intercept is (0, 4); further that the slope is 0. No rise, no run. Therefore, we must graph a line parallel, in this case to the x axis (see graph y = 0x + 4).

When an equation is x = 3, the line parallels the y axis (see graph x = 3).

GRAPH



Systems

When each equation in a system is graphed on the same coordinate axis, either of three possibilities will occur.

1. The lines will intersect and the point of intersection will be the solution to the system. When intersection occurs, the system is called an:

independent system.

2. If there are two parallel lines--no intersection, there is obviously no solution to the system. It is called an:

inconsistent system.

3. If the graphs of the equations are identical--only one line appears (the same line for both equations), it is termed a:

dependent system or coincident system

and has an infinite number of solutions--every point on the line is a solution.

Example 1: Graph the system, $x + 2y = 5$ Equation 1
$2x - y = 5$ Equation 2

Equation 1: $y = -\frac{1}{2}x + \frac{5}{2}$

y-intercept = $\left(0, \frac{5}{2}\right)$

slope = $-\frac{1}{2}$

(See Graph A.)

Equation 2: $y = 2x - 5$

y-intercept = $(0, -5)$

slope = $\frac{2}{1}$

Solution = $(3, 1)$ independent.

Example 2: Graph the system, $4x + 3y = 12$ Equation 1
$8x + 6y = 12$ Equation 2

Equation 1: $y = -\frac{4}{3}x + 3$

y-intercept = (0, 3)

slope = $-\frac{4}{3}$

Equation 2: $y = -\frac{4}{3}x + 2$

y-intercept = (0, 2)

slope = $-\frac{4}{3}$

Note that the slope is the same but two different y intercept (see Graph B).

Solution: $\emptyset$ lines are parallel--inconsistent.

Example 3: Graph the system, $2x + y = 3$ Equation 1
$4x + 2y = 6$ Equation 2

Equation 1: $y = -2x + 3$

y-intercept = (0, 3)

slope = $-\frac{2}{1}$

Equation 2: $y = -2x + 3$

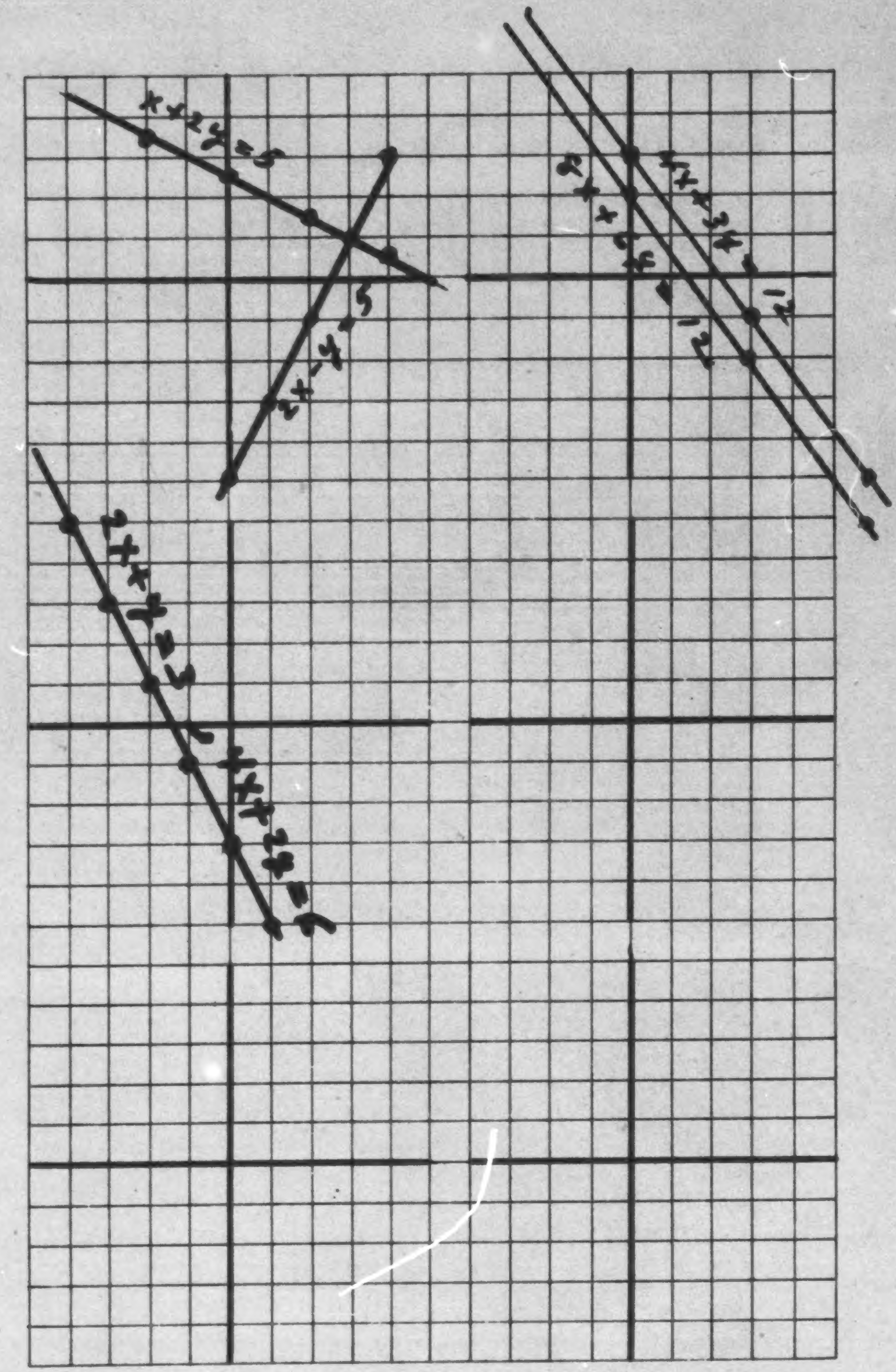
y-intercept = (0, 3)

slope = $-\frac{2}{1}$

Note it is obvious that we have two equations that are identical. Both y intercepts are (0, 3) and both have the same slope (see Graph C).

Solution = infinite number of solutions--dependent.

GRAPHS A, B, AND C



Performance Test

Graph the following functions:

1. $y = x^2$ 2. $y = -2x$

Graph the following quadratic equations:

3. $x^2 + 2x + 1$ 4. $x^2 - 2x - 3$

Graph the following linear equations:

5. $y = -3$ 6. $x - y = 2$

Graph the following systems:

7. $x + y = 4$ Equation 1
$x - y = 2$ Equation 2

8. $2x + y = 3$ Equation 1
$x + 2y = 0$ Equation 2

9. If there is an intersection of lines, the system is called ____________________.

10. If the lines are parallel, the system is ____________________.

11. If both equations of a system graph the same line, the system is ____________________.

Identify:

12. The slope of Equation 1, problem 7.
The y-intercept of Equation 2, problem 7.
The solution set of problem 7.

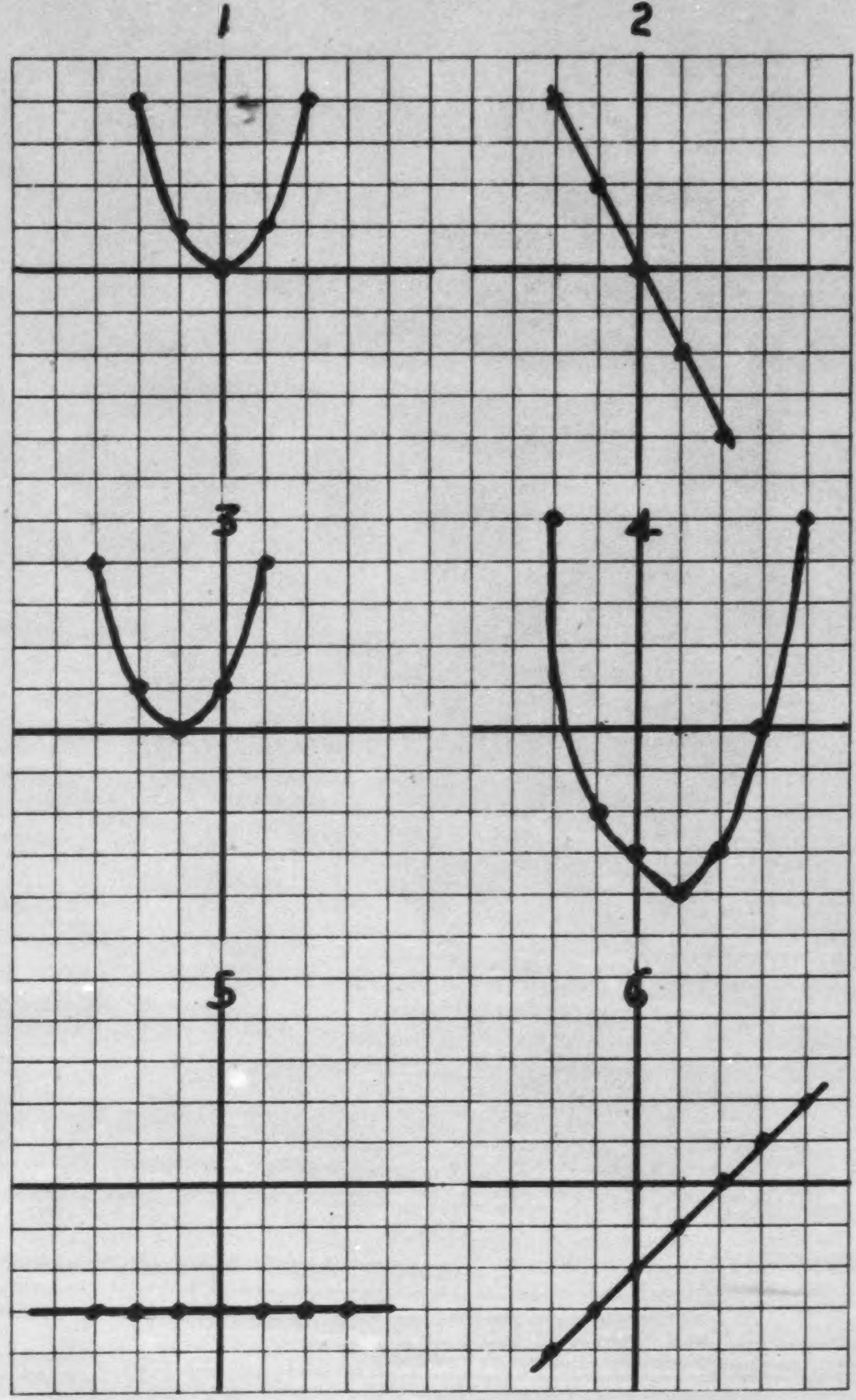
13. The slope of Equation 2, problem 8.
The y-intercept of Equation 1, problem 8.
The solution set of problem 8.

Graph the following lines:

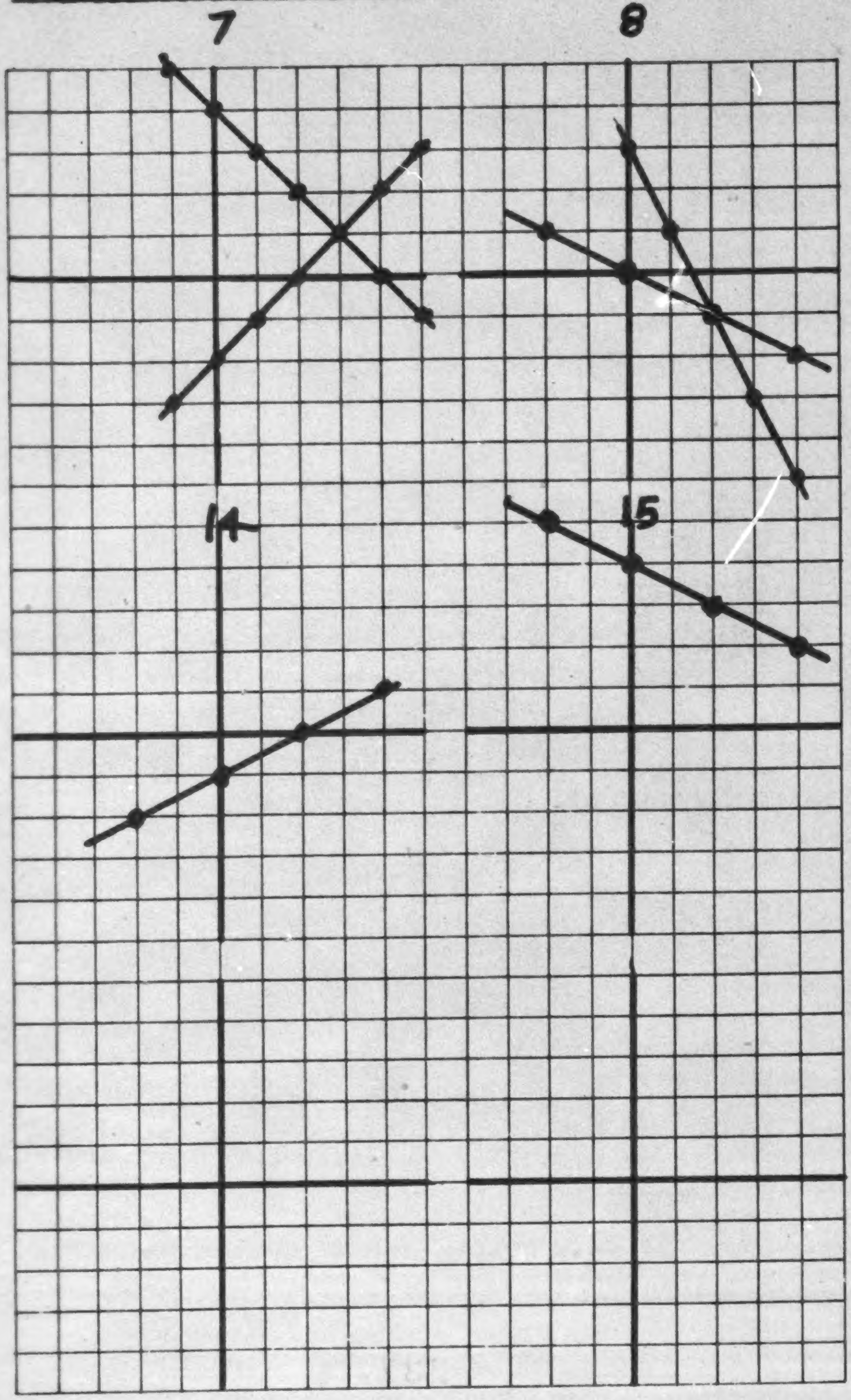
14. point (-2, -2); $m = \frac{1}{2}$

15. y-intercept 4; $m = -\frac{1}{2}$

Performance Test Answer Key



Performance Test Answer Key (Continued)



Performance Test Answer Key (Continued)

9. Independent--a solution at intersection.

10. Inconsistent--∅ no solution.

11. Dependent or coincident--infinite number of solutions.

12.	slope = $-\frac{1}{1}$	Equation 1, problem 7.
	y-intercept = (0, -2)	Equation 2, problem 7.
	Solution Set = (3, 1)	Problem 7.
13.	slope = $-\frac{1}{2}$	Equation 2, problem 8.
	y-intercept = 3	Equation 1, problem 8.
	Solution Set = (2, -1)	Problem 8.

A-PAK #17
ROOTS

Rationale

There are several means of extracting roots. The method selected is related to the reason for extracting the root. A longhand method is practical, a root chart or slide rule is convenient, and a calculator is infallible. Algebraically, however, we are often restricted to simplifying an expression containing a radical sign or perhaps finding approximations like these:

$$\sqrt{2} = 1.414 \qquad \sqrt{3} = 1.732 \qquad \sqrt{5} = 2.236$$
$$= 1.4 \qquad\qquad = 1.7 \qquad\qquad = 2.3$$

Roots also introduce a heretofore unexplored part of the real number system--irrational numbers--numbers that do not repeat or terminate.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Demonstrate proficiency in obtaining roots from a square or cube root table for given radicals.

2. Identify from a set of real numbers irrational numbers.

3. Write the three laws of radicals.

4. Apply the correct law of radicals in order to solve or simplify given expressions.

5. Solve or simplify given radical expressions preceded by a negative sign and/or containing variables.

Performance Activities

1. Attend the lecture on Irrational Numbers.

2. Read "Radicals and Roots" included in this A-PAK.

3. Examine the "Square Root and Cube Root Table," page 408, in Technical Mathematics.

4. Complete the "Root Extraction Exercise" included in this A-PAK.

5. Study pages 48-54 in Technical Mathematics.

6. Practice your skills with the following problem sets: problems 1-45, odd numbers only, page 50; problems 1-27, odd numbers only, page 54.

ROOT EXTRACTION EXERCISE

Extract the approximate roots of the following problems. You may use the Approximate Root Table located on page 408 of __Technical Mathematics__.

$\sqrt[2]{x}$	$\sqrt[3]{x}$	$-\sqrt[2]{x}$
1. $x = 1$	6. $x = 2$	11. $x = 7$
2. $x = 3$	7. $x = -5$	12. $x = 20$
3. $x = 13$	8. $x = 14$	13. $x = 11$
4. $x = 47$	9. $x = -18$	14. $x = 14$
5. $x = 5$	10. $x = 50$	15. $x = 4$

Can you memorize the square root of:

16. 2 =

17. 3 =

18. 5 =

Find $\sqrt{10\,n}$ when:

19. $n = 5$

20. $n = 23$

SOLUTION TO ROOT EXTRACTION EXERCISE

1. 1
2. 1.732
3. 3.606
4. 6.856
5. 2.236
6. 1.260
7. -1.710
8. 2.410
9. -2.621
10. 3.684
11. -2.646
12. -4.472
13. -3.317
14. -3.742
15. -2

16. 1.414
17. 1.732
18. 2.236
19. $\sqrt{50} = 7.071$
20. $\sqrt{230} = 15.166$

RADICALS AND ROOTS

The real number system is composed of:

I. **RATIONAL NUMBERS**

1. Whole numbers and zero.
2. Fractions (decimals).
3. Both + and - whole numbers and fractions.

II. **IRRATIONAL NUMBERS**

1. π
2. Roots like $\sqrt{2}$ or $\sqrt[3]{3}$

Square Root Definition

A positive number contains two square roots, a positive and a negative. For example:

$$\sqrt{81} = +9\cdot+9 \qquad \text{or} \qquad \sqrt{81} = -9\cdot-9$$

Negative numbers do not have square roots although cube roots may be extracted from negative numbers.

$$\sqrt[3]{-27} = -3\cdot-3\cdot-3$$

The square root of a variable is:

$$\sqrt{x^2} = x\cdot x$$

We are not concerned about the negative.

Laws of Radicals

There are three laws of radicals that are applied to solving problems containing roots.

First Law of Radicals: $\sqrt[n]{xy} = \sqrt[n]{x}\cdot\sqrt[n]{y}$

This law permits an algebraic simplification.

$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y}$$

$$\sqrt{45} = \sqrt{9}\cdot\sqrt{5} = 3\sqrt{5}$$

Note that the 45 is factored: $5\cdot3\cdot3$ and $3\cdot3$ is a "perfect square", thus permitting extraction of its roots.

Example 1: Simplify.

$$\begin{aligned}\sqrt{50a^2} &= \sqrt{5}\sqrt{5}\sqrt{2}\sqrt{a}\sqrt{a}\\ &= \sqrt{25}\sqrt{a^2}\sqrt{2}\\ &= 5a\sqrt{2}\end{aligned}$$

Example 2: Simplify.

$$\begin{aligned}-\sqrt{8x^3y^2} &= (-)\sqrt{4}\sqrt{2}\sqrt{x^2}\sqrt{x}\sqrt{y^2}\\ &= -2xy\sqrt{2}\end{aligned}$$

Example 3: Simplify.

$$\sqrt[3]{-56x} = \sqrt[3]{-8}\sqrt[3]{7x} = -2\sqrt[3]{7x}$$

Second Law of Radicals: $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

$$\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$$

Example 1: Simplify.

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3} = \frac{1}{3}\sqrt{6}$$

Note how the $\frac{\sqrt{2}}{\sqrt{3}}$ was multiplied by $\frac{\sqrt{3}}{\sqrt{3}}$ in order to clear the denominator of the radical sign.

Example 2: Simplify.

$$\sqrt[3]{\frac{1}{2}} = \frac{\sqrt[3]{1}}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{1}{2}\sqrt[3]{4}$$

Example 3: Simplify.

$$-\sqrt{\frac{3}{x}} = (-)\frac{\sqrt{3}}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = -\frac{1}{x}\sqrt{3x}$$

Sums of roots may also be simplified.

Example 1: Simplify:

$$3\sqrt{2} - \sqrt{3} + 4\sqrt{2} - 6\sqrt{3} =$$

$$3\sqrt{2} + 4\sqrt{2} - \sqrt{3} - 6\sqrt{3} =$$

$$7\sqrt{2} - 7\sqrt{3} =$$

Example 2: Simplify.

$$4\sqrt{2} - 2\sqrt{2} + \sqrt{2} =$$

$$4 - 2 + 1\sqrt{2} = 3\sqrt{2}$$

Example 3: Simplify.

$$2\sqrt{8} + 5\sqrt{32} =$$

$$2\sqrt{4}\sqrt{2} + 5\sqrt{16}\sqrt{2} =$$

$$2\cdot 2\sqrt{2} + 5\cdot 4\sqrt{2} =$$

$$4\sqrt{2} + 20\sqrt{2} = 4 + 20\sqrt{2} = 20\sqrt{2}$$

Third Law of Radicals. This law is also one of the rules of exponents (Rule VI).

$$x^{\frac{m}{n}} \quad \sqrt[n]{x^m}$$

This law permits a radical to be rewritten as a base with a fractional exponent.

Example 1: $\sqrt[3]{25} = 25^{\frac{1}{3}}$

Example 2: $\sqrt[3]{27^2} = 27^{\frac{2}{3}}$

Performance Test

1. Circle the irrational numbers of the following set:

$1 \qquad 2 \qquad \sqrt{1} \qquad \sqrt{2} \qquad \pi \qquad \sqrt[3]{1}$

$\frac{1}{4} \qquad .506 \qquad \frac{22}{7} \qquad 3.1416 \qquad \sqrt{3} + 1$

2. Find the roots of the following by means of the square root table:

(a) $\sqrt{2}$ (d) $\sqrt[3]{14}$

(b) $\sqrt{3}$ (e) $-\sqrt{7}$

(c) $\sqrt{5}$ (f) $-\sqrt{4}$

3. Write the three laws of radicals.

4. Simplify the following:

(a) $\sqrt{12} =$ (f) $\sqrt{63x^7}$

(b) $-\sqrt{\frac{1}{49}} =$ (g) $\frac{2\sqrt{5} - \sqrt{10}}{\sqrt{5}} =$

(c) $3\sqrt{5} \cdot 4\sqrt{5}$ (h) $3\sqrt{8} + \sqrt{2}$

(d) $\sqrt{-x^5}$ (i) $\sqrt[3]{-27x^3} =$

(e) $\frac{\sqrt{7}}{\sqrt{8}} =$ (j) $\frac{2 + \sqrt{5}}{1 - \sqrt{5}} =$

Performance Test Answer Key

1. $\sqrt{2}$ π $\sqrt[3]{3} + 1$

2. (a)	1.414	(d)	2.410
(b)	1.732	(e)	-2.646
(c)	2.236	(f)	-2

3. **First Law:** $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Second Law: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Third Law: $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

4. (a) $\sqrt{12} = \sqrt{3}\sqrt{4} = 2\sqrt{3}$

(b) $-\sqrt{\frac{1}{49}} = -\frac{\sqrt{1}}{\sqrt{49}} = -\frac{1}{7}$

(c) $3\sqrt{5}\,4\sqrt{5} = 12\sqrt{5}^2 = 12 \cdot 5 = 60$

(d) $-\sqrt{-x^5} = -\sqrt{x^4}\ \sqrt{-x} = -x^2\sqrt{-x}$

(e) $\frac{\sqrt{7}}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{56}}{8} = \frac{1}{8}\sqrt{4}\sqrt{4}\sqrt{4} = \frac{1}{8} \cdot 2 \cdot 2 \cdot 2 = 1$

(f) $\sqrt{63x^7} = \sqrt{7x}\sqrt{9x^6} = 3x^3\sqrt{7x}$

(g) $\frac{2\sqrt{5} - \sqrt{10}}{\sqrt{5}} = \frac{2\sqrt{5} - \sqrt{2}\sqrt{5}}{\sqrt{5}} = 2 - \sqrt{2}$

(h) $3\sqrt{8} + \sqrt{2} = 3\sqrt{2}\sqrt{4} + \sqrt{2} = 6\sqrt{2} + \sqrt{2} =$
$\sqrt{2}(6 + 1) = 7\sqrt{2}$

(i) $\sqrt[3]{-27x^3} = -3x$

(j) $\frac{2 + \sqrt{5}}{1 - \sqrt{5}} \cdot \frac{1 + \sqrt{5}}{1 + \sqrt{5}} = \frac{7 + 3\sqrt{5}}{-4} = \frac{7 + 3\sqrt{5}}{4}$

A-PAK #18
WORD PROBLEMS

Rationale

Word problems represent a culmination of learning algebra and intuitive reasoning. Therefore, solving typical problems has been reserved for the final algebraic packet. It is important to approach word problems with a predetermined system for finding their solution. A positive attitude and determination will facilitate finding solutions.

Instructional Objectives

After completing this A-PAK, the learner will be able to:

1. Develop a personalized system for solving word problems.

2. Solve open sentences when given equations are supplied.

3. Write open sentences for given word problems.

4. Write open inequalities for given word problems.

5. Write a personal opinion explaining the confidence or lack of confidence about your capability concerning word problems.

Performance Activities

1. Attend the lecture on Writing Open Sentences.

2. Read "Applications - Open Sentences" included in this A-PAK.

3. Review the slide presentation "Open Sentences," Eye-Gate 5-6A, at the Center for Independent Study.

4. Study pages 214-215 in Technical Mathematics.

5. Practice your skills with the following problem sets: Exercise 46, page 217, problems 1 through 30.

6. Consult the textbook or books required in your major study area. Compile a file of sample word problems and submit for "extra credit."
   (a) Print the word problem on a 3 x 5 card. Show the source of the problem.
   (b) Present the solution to the problem on the opposite side of the card.

APPLICATIONS - OPEN SENTENCES

One practical application of algebra is the solving of word problems. Generally the problem can be solved by substituting problem data for variables (providing the equation is known) and then mechanically solving the equation.

Example 1: Given a cutting speed formula, determine the RPM of a twist drill.

CS = cutting speed in feet per minute = 80.
RPM = revolutions per minute of the drill.
d = diameter of the drill = $\frac{3}{8}$"

$$RPM = \frac{4CS}{d}$$

Substitute: $$RPM = \frac{4(80)}{\frac{3}{8}}$$

Solve: $$RPM = 4 \cdot 80 \cdot \frac{8}{3} = \frac{2560}{3} = 853$$

$$RPM = 853$$

Example 2: Determine the length of one of the sides of a right angle triangle given the other two sides, utilizing Pythagorean theorem.

a is unknown; b = 3"; c = 5"

$$a^2 + b^2 = c^2$$

$$a^2 = c^2 - b^2$$

$$a = \sqrt{c^2 - b^2}$$

Substitute: $$a = \sqrt{(5)^2 - (3)^2} = \sqrt{25 - 9} = \sqrt{16}$$

$$b = 4$$

Example 3: Find the circular cross section of a conductor (wire) utilizing the formula:

A = circular cross section.
$\pi = \frac{22}{7}$
d = diameter of wire .025 or $\left(\frac{25}{1000} = \frac{1}{40}\right)$

$$A = \frac{\pi d^2}{4}$$

Substitute: $$A = \frac{22}{7} \cdot \left(\frac{1}{40}\right)^2 \cdot \frac{4}{1}$$

$$A = \frac{22}{7} \cdot \frac{1}{1600} \cdot \frac{4}{1} = \frac{22}{2800}$$

$$A = \text{approx. } .007857$$

Variables

A variable may represent members of a set called the domain or replacement set of the variable. The variable is a symbol. A variable with just one value is termed a constant.

An equation like $x + 7 = 14$ or an inequality such as $x + 7 < 16$ contains a variable and is called an open sentence. In the first example, the domain is $\{8\}$. The set of numbers for which an open sentence is true is called the solution set. For the inequality $x + 7 < 16$ the solution set is

{all numbers less than nine}

When the data is presented but the formula is not available, an equation must be constructed. This is called "writing an open sentence." There are several types of traditional algebra word problems; however, only a few are presented as examples. The learner should try to develop an orderly method of problem solving similar to the following:

1. Read the problem several times.

2. Note what information (data) is supplied.

3. Select representation symbols (variables) for the unknown quantity or quantities.

4. A drawing, chart, or visual sketch will often suggest relationships.

5. Perhaps writing the open sentence in words may help--especially at first.

Integer Problems

Example 1a: Find three consecutive odd integers whose sum is 98.

Representation:

x = 1st odd integer
$x + 2$ = 2nd odd integer
$x + 4$ = 3rd odd integer

Open Sentence: $x + (x + 2) + (x + 4) = 98$

Solve:

$$\begin{aligned} x + x + 2 + x + 4 &= 98 \\ 3x + 6 &= 98 \\ 3x &= 93 \\ x &= 31 \end{aligned}$$

Solution:

1st integer $x = 31$
2nd integer $x + 2 = 33$
3rd integer $x + 4 = 35$

Example 1b: The difference between two numbers is 9. Their sum is 21. Find both numbers.

Representation:

x = 1st number
y = 2nd number

Open Sentences: $y - x = 9$ and $y + x = 21$

Solve:

$$\begin{array}{r} y - x = 9 \\ \underline{(y + x = 21)} \end{array} \quad = \quad \begin{array}{r} y - x = 9 \\ \underline{-y - x = -21} \\ -2x = -12 \\ x = 6 \end{array}$$

$$\begin{aligned} y - x &= 9 \\ y - 6 &= 9 \\ y &= 15 \end{aligned}$$

Solution:

$x = 6$
$y = 15$

Example 1c: Three times a given number is greater than the number plus 8. Find the given number.

Representation: x = given number

Open inequality: $3x > x + 8$

Solve: $3x - x > 8$

$2x > 8$

$x > 4$

Solution: $x > 4$

Example 1d: Find three consecutive integers if the sum of the first and third is 98.

Representation: 1st integer $= x$
2nd integer $= x + 1$
3rd integer $= x + 2$

Open Sentence: $x + x + 2 = 98$

Solve: $2x = 96$

Solution: $x = 48$

1st integer $x = 48$
2nd integer $x + 1 = 49$
3rd integer $x + 2 = 50$

Geometric Problems

Example 2a: The perimeter of a square is 32 inches. Determine the length of each side.

Representation: x = length of one side.

Open Sentence: $4x = 32$

Solve: $4x = 32$

$x = 8$

Solution: length of side = 8

Example 2b: The width of a rectangle is 8 inches less than its length. The perimeter is 96 inches. Determine the length and the width.

Representation: x = length

$x - 8$ = width

Open Sentence: $2(x - 8) + 2x = 96$

Solve:

$$2(x - 8) + 2x = 96$$

$$2x - 16 + 2x = 96$$

$$4x = 112$$

$$x = 28$$

Solution: length = 28

width = 28 - 8 = 20

Example 2c: The perimeter of a rectangle is 28 metres and the area is 40 square metres. Determine the length and width of the rectangle.

Representation: L = length; w = width

$$2L + 2w = 28$$

$$L + w = 14$$

Representation: Let length = L; width = $14 - L$

Open Sentence: $L(14 - L) = 40$

Solve:

$$-L^2 + 14L - 40 = 0$$

$$L^2 - 14L + 40 = 0$$

$$(L - 10)(L - 4) = 0$$

$$L = 10; L = 4$$

Solution: $L = 10$; $w = 4$ (reject $L = 4$)

Age Problems

Example 3a: John is 18 years older than Don. Together their ages are 46 years. How old is each?

Representation: Don $= x$; John $= x + 18$

Open Sentence: $x + x + 18 = 46$

Solve: $2x = 28$

$x = 14$

Solution: Don $= 14$; John $= 14 + 18 = 32$

Example 3b: Barbara is 24 years older than Leslie. In 2 years Barbara will be twice Leslie's age. How old is each?

Representation: Leslie $= x$; Barbara $= x + 24$

Open Sentence: $2(x + 2) = (x + 24) + 2$

Solve: $2x + 4 = x + 26$

$x = 22$

Solution: Leslie $= 22$; Barbara $= 22 + 24 = 46$

Example 3c: Dorothy is 3 times Sue's age. Three years ago the sum of their ages was 26. How old was Sue at that time?

Representation: Sue $= x$; Dorothy $= 3x$

Open Sentence: $(x - 3) + (3x - 3) = 26$

Solve: $x - 3 + 3x - 3 = 26$

$4x = 32$

$x = 8$

Solution: Sue's age was 8 (three years ago).

Mixture Problems

Example 4a: How much 70 percent by volume sulfuric acid must be added to 5 quarts of 40 percent pickling solution in order to increase its strength to 60 percent?

Representation: x = number of quarts 68 percent solution
$x + 5$ = number of quarts final solution

Open Sentence: $.40(5) + .70(x) = .60(x + 5)$

Solve:

$$2x + .70x = .60x + 3.40$$

$$200 + 70x = 60x + 340$$

$$10x = 140$$

$$x = 10$$

Solution: 10 quarts of 68 percent solution

Example 4b: A 70 pound alloy containing 92 percent iron must be compounded. How much steel containing 60 percent iron and how much scrap iron containing 98 percent iron must be mixed?

Representation: x = lbs. of steel
$70 - x$ = lbs. of scrap iron

Open Sentence: $.60x + .98(70 - x) = .92(70)$

Solve:

$$.60x + 68.60 - .98x = 64.40$$

$$60x + 6860 - 98x = 6440$$

$$-38x = -420$$

$$x = 11.3+$$

Solution: 11.3+ lbs. steel; $70 - 11.3 = 58.7$ lbs. scrap iron

Example 4c: How many ounces of imported spice worth 1.25 cents per ounce must be added to 14 ounces of domestic spice worth 70 cents per ounce in order to form a mixture worth 90 cents per ounce?

Representation: x = number of ounces of imported spice.

Open Sentence: $1.25x + .70(14) = .90(x + 28)$

Solve:

$$1.25x + 9.80 = .90x + 25.20$$

$$125x + 980 = 90x + 2520$$

$$35x = 1440$$

$$x = 41+$$

Solution: 41+ ounces of imported spice must be added.

Motion Problems

Motion problems utilize the relationship:

distance = rate x time; $d = r \cdot t$

Example 5a: A vacationer takes 3 hours less time returning home than he did on his outbound trip. If his average speed was 45 mph outbound and 54 mph returning, what was his return time?

Chart:

	d	=	r	·	t
out	$45x$		45		x
back	$54(x - 3)$		54		$x - 3$

Representation: x = time out
$x - 3$ = time back

Open Sentence: $45x = 54(x - 3)$

Solve:

$$45x = 54x - 162$$

$$-9x = -162$$

$$x = 18$$

Solution: Time back $18 - 3 = 15$

Example 5b: Two algebra students leave from the same location but travel in opposite directions. One of the students travels 3 mph faster than the other. At the end of 6 hours, they are 498 miles apart. At what rate was each traveling?

Chart:

$d =$	$r \cdot$	t
$498 =$	x	6
	$x + 3$	6

Open Sentence: $6x + 6(x + 3) = 498$

Solve:

$$6x + 6x + 18 = 498$$

$$12x = 480$$

$$x = 40$$

Solution: One student 40 mph
Other student $40 + 3 = 43$ mph

Example 5c: Two automobiles are heading toward each other and are 130 miles apart. If the rate of one automobile is 4 mph greater than the rate of the other automobile and they rendezvous in 5 hours, what is the rate of each automobile?

Chart:

	$d =$	$r \cdot$	t
1st auto	$5x$	x	5
2nd auto	$5(x + 4)$	$x + 4$	5

Open Sentence: $5x + 5(x + 4) = 130$

Solve:

$$5x + 5x + 20 = 130$$

$$10x = 110$$

$$x = 10$$

Solution: Rate of 1st auto = 10 mph.
Rate of 2nd auto = 14 mph.

Performance Test

A performance test will be distributed by the instructor along with the performance test answer key.

Both the performance test and the answer key will be compiled from word problems submitted by the students (one of the performance activities). A few problems from a noted algebra text are included as examples; however, they may not be duplicated for distribution since they are protected under copyright.

Sample Performance Test (not for distribution)

Write an open sentence or open inequality and then solve the following word problems.* Utilize your personalized system--an orderly and consistent method.

Integer

P. 119 1. "Find four consecutive integers if the sum of the second and fourth is 58."

P. 120 2. "Find the least consecutive odd integers whose sum is greater than 16."

Motion

P. 123 3. "Marie rode her bicycle from her home to the bicycle shop in town and then walked back home. If she averaged 6 miles per hour riding and 3 miles per hour walking, how far is it from her home to the bicycle shop if her traveling time totaled 1 hour?"

p. 124 4. "Two ships are sailing toward each other and are 120 nautical miles apart. If the rate of one ship is 4 knots greater than the rate of the other, and if they meet in 3 hours, find the rate of each ship."

Mixture

p. 126 5. "How many pounds of candy worth 90¢ per pound must be added to 10 pounds of a candy worth 60¢ per pound to form a mixture worth 80¢ per pound?"

p. 127 6. "How many pounds of coffee worth 50¢ per pound must be added to 15 pounds of coffee worth 80¢ a pound to form a mixture worth 60¢ per pound?"

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*These word problem samples are selected from Elementary Algebra for College Students, Mary Dolciani and Robert Sorgenfrey, Houghton Mifflin Co., New York, 1971.

Sample Performance Test Answer Key

1. Representation: $x, x + 1, x + 2, x + 3$

   Open Sentence: $x + 1 + x + 3 = 58$

   Solution: 27, 28, 29, 30

2. Representation: $x, x + 2$

   Open Inequality: $x + x + 2 \quad 16$

   Solution: $x \quad 7$, i. e., 8, 9, 10

3. Chart:

	c	·	t	=	d
c	4		$t + 4$		$4(t + 4)$
m	12		t		$12t$

   Open Sentence: $4(4 + 4) = 12t$

   Solution: $2 = t$

4. Chart:

	r	·	t	=	d
1st	x		3		$3x$
2nd	$x + 4$		3		$3(x + 4)$

   Open Sentence: $3x + 3(x + 4) = 120$

   Solution: $x = 16$ (16 knots, 20 knots)

5. Representation: 1st kind of candy = $.90x$
   2nd kind of candy = $.60(10)$
   mix = $10 + x = .80(10 + x)$

   Open Sentence: $.90x + 6.00 = .80(10 + x)$

   Solution: $x = 20$

6. Representation: 1st kind of coffee = $.50x$
   2nd kind of coffee = $.80(50)$
   mix = $.60(15 + x)$

   Open Sentence: $.50x + .80(50) = .60(15 + x)$

   Solution: $x = 30$

APPENDIX B "HUNDRED-PROBLEM ARITH. TEST"
AND APPENDIX C "FORM A, ALGEBRA I, COOPERATIVE
MATHEMATICS TESTS" WERE REMOVED DUE TO
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APPENDIX D

Traditional Class____________

A-PAK Class__________________

STUDENT EVALUATION QUESTIONNAIRE

This questionnaire will help in the assessment of the A-PAK treatment as compared to a traditional algebra treatment. Please answer every statement by circling + if you agree, - if you disagree, or o if you have no opinion.

+ - o 1. The algebra section helped me feel confident as I was taking various course tests.

+ - o 2. The algebra section of the course was an interesting part of Technical Math.

+ - o 3. I would like to learn more algebra.

+ - o 4. The algebra section contained challenging as well as a variety of learning activities.

+ - o 5. Learning objectives were clearly identified. I understood what I was supposed to learn.

+ - o 6. The mathematical exercises and/or activities were practical; they assisted in my understanding of the material.

+ - o 7. I did not experience difficulty locating recommended study materials.

+ - o 8. The algebra section was generally understandable (+), confusing (-), no opinion (o).

+ - o 9. The course treatment of algebra was (for me) a convenient way to learn.

+ - o 10. This was a good (+), bad (-), average (o) treatment of algebra.

+ - o 11. I would recommend this course to other students.

+ - o 12. The algebra lectures, chalkboard talks, and discussions assisted me during the course.

+ - o 13. The Center for Independent Study was of assistance to me.

+ - o 14. The audiovisual, games, special treatments, etc., provided more motivation.

+ - o 15. In total, I would rate this an educational experience (+), a waste of time (-), not worth responding to (o).